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Autores / Autors   Galindo Pastor, Jorge ; Tkachenko, Mikhail ; Bruguera, Montserrat ; Hernández, Constancio

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REFLEXIVITY IN PRECOMPACT GROUPS AND EXTENSIONS

J. GALINDO, M. TKACHENKO, M. BRUGUERA, AND C. HERNÁNDEZ

ABSTRACT. We establish some general principles and find some counter-examples concerning the Pontryagin reflexivity of precompact groups and P-groups. We prove in particular that:

(1) A precompact Abelian group G of bounded order is reflexive iff the dual group $G^\wedge$ has no infinite compact subsets and every compact subset of G is contained in a compact subgroup of G.

(2) Any extension of a reflexive P-group by another reflexive P-group is again reflexive.

We show on the other hand that an extension of a compact group by a reflexive ω-bounded group (even dual to a reflexive P-group) can fail to be reflexive.

We also show that the P-modification of a reflexive σ-compact group can be non-reflexive (even if, as proved in [20], the P-modification of a locally compact Abelian group is always reflexive).

1. INTRODUCTION

The papers [1, 19, 20, 21] have unveiled that the duality properties of the class of precompact groups are more complicated than expected. The following theorem summarizes

some of the known facts that concern the duality of precompact groups, see below for unexplained terminology.

Theorem 1.1. *Let G be an Abelian group and let τ_H be a precompact topology on G induced by some group of homomorphisms $H \subset \mathrm{Hom}(G, \mathbb{T})$. The topological group $(G, \tau_H)^\wedge$ dual to the precompact group (G, τ_H) can be:*

(1) *a discrete group, as for instance when H is countable (see [3] and [9]) or (G, τ_H) is the Σ-product of uncountably many copies of the discrete group $\mathbb{Z}(2)$ (this can be deduced from (the proof of) Lemma 27.11 of [5], see Lemma 5.2 below). In this case (G, τ_H) is not reflexive.*

(2) *a nondiscrete P-group. This is the case when when (G, τ_H) is the ω-bounded group that arises as the dual of a reflexive P-group, as those constructed in [20] and [21]. Obviously (G, τ_H) is reflexive in this case.*

(3) *a precompact noncompact group, as is the case of the infinite pseudocompact groups with no infinite compact subsets constructed in [19] and [1]. These groups are reflexive.*

(4) *a compact group, as happens when $H = \mathrm{Hom}(G, \mathbb{T})$, the family of all homomorphisms of G to $\mathbb{T}$.*

The bases on which the reflexivity of precompact groups stands remain elusive so far. In this paper we give a first insight to this issue by establishing some general facts and giving some counterexamples to what could be regarded as reasonable generalizations of known results. We prove in Proposition 2.10 that a precompact Abelian group of bounded order is reflexive if and only if the compact subsets of the dual group $G^\wedge$ are finite and every compact subset of G is contained in a compact subgroup.

We will especially address the behavior of reflexivity under extensions in precompact, ω-bounded, and P-groups. We recall that the class of P-groups is naturally linked to that of precompact groups through duality since, by [20, Lemma 4.1], the dual group $G^\wedge$ of every P-group G is ω-bounded and hence precompact.

Our starting point in this regard is the fact that an extension of a reflexive group by a compact group is again reflexive provided that the dual of the extension separates points of the extension (see [6, Theorem 2.6]). We show in Example 4.3 that an extension of a compact group by a reflexive precompact (even ω-bounded) group may be non-reflexive. In Corollary 3.5 we prove, in contrast, that an extension of a reflexive P-group by another reflexive P-group is always reflexive.

Answering a question in [20] we prove in Section 5 that, while the P-modification of an LCA group is always reflexive, the P-modification of a reflexive σ-compact group can fail to be reflexive.

1.1. **Notation and terminology.** All groups considered here are assumed to be Abelian. A *character* of a topological group G is a continuous homomorphism of G to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ when the latter is considered as a subgroup of in the complex plane $\mathbb{C}$ with its usual topology and multiplication. The group $G^\wedge$ of all characters of G with the pointwise multiplication is called the *dual group* or simply the *dual* of G. The dual group $G^\wedge$ carries the *compact-open* topology τ_{co} defined as follows.

Put $\mathbb{T}_+ = \{z \in \mathbb{T} : Re(z) \geq 0\}$. For a nonempty set $K \subset G$, we define

$$K^\triangleright = \left\{ \chi \in G^\wedge \colon \chi(K) \subset \mathbb{T}_+ \right\}.$$

The collection of sets $\{K^\triangleright \colon K \subset G, \ K \text{ is compact}\}$ forms a neighborhood basis at the identity of $G^\wedge$ for the compact-open topology τ_{co}. Let us note that the sets $K^\triangleright$ are not necessarily open in $(G^\wedge, \tau_{co})$ since $\mathbb{T}_+$ is not open in $\mathbb{T}$. If, instead of $\mathbb{T}_+$, we use a smaller neighborhood U of 1 in $\mathbb{T}$ to construct the sets $K^\triangleright$, the resulting sets will also form a neighbourhood basis at the identity for the compact-open topology τ_{co}.

The subgroup

$$K^\perp = \left\{ \chi \in G^\wedge \colon \chi(K) = \{1\} \right\}$$

of $G^\wedge$ is called the *annihilator* of a set $K \subset G$. If $B \subset G^\wedge$, we will also find useful to refer to the set $B^\triangleleft = \{x \in G \colon \chi(x) \in \mathbb{T}_+ \text{ for all } \chi \in B\}$.

A subset A of a topological group G is called *quasi-convex* if $A = (A^{\triangleright})^{\triangleleft}$. If quasi-convex sets in G form a neighborhood base of the neutral element in G, the group G will be called *locally quasi-convex* [3].

The *bidual group* of G is $G^{\wedge\wedge} = (G^{\wedge})^{\wedge}$. The *evaluation mapping* $\alpha_G \colon G \to G^{\wedge\wedge}$ is defined by

$$\alpha_G(x)(\chi) = \chi(x),$$

for all $x \in G$ and $\chi \in G^{\wedge}$. It is easy to see that α_G is a homomorphism. If it is a topological isomorphism of G onto $G^{\wedge\wedge}$, the group G is called *reflexive*. Every reflexive group is locally quasi-convex [3, Proposition 6.6].

Let H be a subgroup of a topological group G. We say that H *dually embedded* in G if each continuous character of H can be extended to a continuous character of G. A subgroup H of a topological group G is said to be *h-embedded* into G provided that any homomorphism φ of H to an arbitrary compact group K is extendable to a continuous homomorphism $\tilde{\varphi} \colon G \to K$. Note that if H is an h-embedded subgroup of G, then any homomorphism of H to a compact group is continuous. Note also that every h-embedded subgroup H of G is dually embedded in G.

Let N be a closed subgroup of a topological group G. The group G is usually called an *extension* of G/N by the group N. For example, every feathered (equivalently, almost metrizable) Abelian group is an extension of a metrizable group by a compact group (see [2, Theorem 4.3.20]).

A space X is called ω-*bounded* provided the closure of every countable subset of X is compact. It is clear that every ω-bounded space is countably compact, but not vice versa.

A Tychonoff space is *pseudocompact* if every continuous real-valued function on the space is bounded. According to [14, Theorem 1.2], a precompact topological group H is pseudocompact iff G intersects every non-empty G_δ-set in the Raĭkov completion ϱH of H. Every pseudocompact topological group is precompact [14, Theorem 1.1]. Hence all ω-bounded and countably compact groups are precompact. By a well-known theorem

of Comfort and Ross in [13], a topological group (G, τ) is precompact if and only if the topology τ of G is the topology τ_H generated by a group of characters $H \subset \text{Hom}(G, \mathbb{T})$.

By a *protodiscrete* group we understand a topological group having a basis of neighborhoods of the identity consisting of open subgroups. Protodiscrete groups are also known as linear groups. Evidently, protodiscrete Abelian groups are locally quasi-convex.

A *P-space* is a space in which every G_δ-set is open. A *P-group* is a topological group which is a *P*-space. According to [2, Lemma 4.4.1], every *P*-group is protodiscrete.

Given a space X, the *P-modification* of X, denoted by PX, is the underlying set X endowed with the topology whose base consists of G_δ-sets in the original space X. It is clear that the *P*-modification of a topological group is again a topological group.

Let A denote a non-empty index set and let, for each $\alpha \in A$, G_α be a compact group with identity e_α. Given $x \in \prod_{\alpha \in A} G_\alpha$, we put

$$\text{supp}(x) = \{\alpha \in A : x_\alpha \neq e_\alpha\},$$

$$\sum \prod_{\alpha \in A} G_\alpha = \{x \in \prod_{\alpha \in A} G_\alpha : |\text{supp}(x)| \leq \omega\},$$

and

$$\bigoplus_{\alpha \in A} G_\alpha = \{x \in \prod_{\alpha \in A} G_\alpha : |\text{supp}(x)| < \omega\}.$$

It is clear that $\sum \prod_{\alpha \in A} G_\alpha$ is a dense ω-bounded subgroup of $\prod_{\alpha \in A} G_\alpha$ (see [2, Proposition 1.6.30]). In particular, the group $\sum \prod_{\alpha \in A} G_\alpha$ is countably compact. This subgroup of $\prod_{\alpha \in A} G_\alpha$ is called the Σ-*product* of the family $\{G_\alpha : \alpha \in A\}$. When $G_\alpha = G$ for all $\alpha \in A$, we use the symbol ΣG^A instead of $\sum \prod_{\alpha \in A} G_\alpha$. The group $\oplus_{\alpha \in A} G_\alpha$ is usually known as the *direct sum* (and also as the σ-*product*) of the family $\{G_\alpha : \alpha \in A\}$.

2. On the duality of precompact Abelian groups

In this section we collect several general results concerning the duality theory of precompact groups. Some of them appear in the literature, while some others might be known to specialists. It seems nonetheless convenient to have them collected here, as they prove to

be useful in later sections (Corollary 2.11, for instance, is essential in the proof of Example 4.3, one of the main results of the paper).

Lemma 2.1. *If the family $G^\wedge$ separates elements of the group G, then $\alpha_G \colon G \to G^{\wedge\wedge}$ is a monomorphism.*
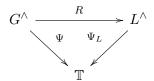
Proof. If $g \in G$ and $g \neq 0_G$, then there exists $\chi \in G^\wedge$ such that $\chi(g) \neq 1$ or, equivalently, $\alpha_G(g)(\chi) \neq 1$. Hence $\alpha_G(g)$ is distinct from the neutral element of $G^{\wedge\wedge}$ and α_G is a monomorphism. $\square$

Lemma 2.2. *Every subgroup L of a precompact group G is dually embedded.*

Proof. Let ϱG be the Weil completion of G and H be the closure of L in ϱG. Then H is a closed subgroup of the compact Abelian group ϱG, so [2, Proposition 9.6.2] implies that H is dually embedded in ϱG. By [2, Proposition 3.6.12], the dense subgroup L of H is dually embedded in H. Hence L is dually embedded in ϱG and in G. $\square$

Proposition 2.3. *Let G be a topological group. If each compact subset of G is contained in a reflexive, dually embedded subgroup, then α_G is onto.*

Proof. Let $\Psi \in G^{\wedge\wedge}$ be an element of the bidual group. Then there is a compact subset $K \subset G$ such that $K^\triangleright \subset \Psi^{-1}(\mathbb{T}_+)$. By hypothesis there is a reflexive dually embedded subgroup L of G with $K \subset L$. Since $L^\perp = L^\triangleright \subset K^\triangleright$ we have that $L^\perp \subset \ker \Psi$. Since L is dually embedded, this implies that Ψ factorizes through $\widehat{L}$. In other words, there is a continuous homomorphism $\Psi_L \colon \widehat{L} \to \mathbb{T}$ that, denoting by R the canonical restriction map of $G^\wedge$ to $L^\wedge$, makes the following diagram commute.

$$
\begin{array}{ccc}
G^\wedge & \xrightarrow{\quad R \quad} & L^\wedge \\
& \searrow{\scriptstyle \Psi} \quad \swarrow{\scriptstyle \Psi_L} & \\
& \mathbb{T} &
\end{array}
$$

Notice that R is a continuous surjective homomorphism, while the continuity of Ψ_L follows from the inclusion $K \subset L$. Since L is reflexive, $\Psi_L = \alpha_L(g)$ for some $g \in L$. This means that $\Psi = \alpha_G(g)$. Indeed, if $\chi \in G^\wedge$ then $\Psi(\chi) = \Psi_L(\chi{\restriction}_L) = \chi{\restriction}_L(g) = \chi(g)$. $\square$

The applicability of Proposition 2.3 in our context is enhanced by the following fact.

Lemma 2.4 (S. Hernández and S. Macario, [24])**.** *If G is a pseudocompact group, then $G^\wedge$ has no infinite compact subsets.*

Proof. Let H be a subgroup of $\mathrm{Hom}(G, \mathbb{T})$ that induces the topology of G. By [24, Proposition 3.4], every countable subgroup of $(H, \mathcal{T}_G)$ carries the finest precompact topological group topology, where $\mathcal{T}_G$ is the topology (on H) of pointwise convergence on elements of G. This implies that $(H, \mathcal{T}_G)$ cannot contain infinite compact subsets, see [17]. The same is true, *a fortiori*, for $G^\wedge$. $\square$

Corollary 2.5. *Let G be a pseudocompact group. If every compact subset of G is contained in a compact subgroup of G, then G is reflexive.*

Proof. Since G is pseudocompact (hence precompact), $G^\wedge$ separates elements of G. Therefore, Lemma 2.1 shows that α_G is a monomorphism. By Lemma 2.2, all subgroups of G are dually embedded. Hence Proposition 2.3 implies that α_G is a group isomorphism. Since, by Lemma 2.4, $G^{\wedge\wedge}$ carries the topology of pointwise convergence on elements of $G^\wedge$, just as G does, this isomorphism is easily seen to be a homeomorphism. The group G is therefore reflexive. $\square$

Proposition 2.6. *If G is precompact and α_G is onto, then every closed metrizable subgroup of G is compact.*

Proof. Let N be a metrizable subgroup of G. If ϱN denotes the completion of N we have, as a consequence of the Auenhofer–Chasco theorem (see [3] or [9]) that $N^{\wedge\wedge} = \varrho N$.

If some closed metrizable subgroup N of G is not compact, there is $\Psi \in N^{\wedge\wedge}$ with $\Psi \notin \alpha_N(N)$. Define now $\Psi^G \colon G^\wedge \to \mathbb{T}$ by $\Psi^G(\chi) = \Psi(\chi{\restriction}_N)$ for each $\chi \in G^\wedge$. Then $\Psi^G \in G^{\wedge\wedge}$. Since α_G is onto by hypothesis, there must be $g \in G$ with $\Psi^G = \alpha_G(g)$. As N is closed it follows that $g \in N$, for otherwise there is $\chi \in G^\wedge$ with $\chi{\restriction}_N = 1$ and $\chi(g) \neq 1$ yielding that $\chi(g) = \alpha_G(g)(\chi) = \Psi^G(\chi) = \Psi(\chi{\restriction}_N) = 1$. But, since $g \in N$, this implies $\Psi = \alpha_N(g) \in \alpha_N(N)$, against our choice of Ψ. $\square$

Proposition 2.7. *Let G be a precompact topological group such that $G^\wedge$ is a protodiscrete group with no infinite compact subsets. Then G is reflexive if and only if every compact subset of G is contained in a compact subgroup of G.*

Proof. Observe that the general hypothesis implies that α_G is a topological isomorphism of G onto a subgroup of $G^{\wedge\wedge}$.

Sufficiency: If every compact subset of G is contained in a compact subgroup of G, then α_G is onto by Lemma 2.3.

Necessity: Suppose that G is reflexive and let K be a compact subset of G. Since $K^\triangleright$ is a neighbourhood of the identity of $G^\wedge$, there is an open subgroup N of $G^\wedge$ with $N \subset K^\triangleright$. Now $\alpha_G(K) \subset K^{\triangleright\triangleright} \subset N^\perp$ and $N^\perp$ is a compact subgroup of $G^{\wedge\wedge} = G$, for N is open in $G^\wedge$ (see [27, Lemma 2.2]). Thus K is contained in the compact subgroup $\alpha_G^{-1}(N^\perp)$ of G. □

Corollary 2.8. *Let G be a precompact topological group such that $G^\wedge$ is a P-group. Then G is reflexive if and only if every compact subset of G is contained in a compact subgroup of G.*

Proof. Clearly all compact subsets of a P-group are finite. Since every P-group is protodiscrete [2, Lemma 4.4.1], the conclusion follows from Proposition 2.7. □

The next Lemma can also be deduced from Proposition 2.1 of [4].

Lemma 2.9. *If G is a precompact group of bounded order, then $G^\wedge$ is a protodiscrete group.*

Proof. Let n be the exponent of G. Choose a neighbourhood V of 1 in $\mathbb{T}$ not containing n-roots of 1 other than 1 itself. Then the equality $\{\chi \in G^\wedge \colon \chi(K) \subset V\} = K^\perp$ holds for every compact set $K \subset G$. These sets form a basis of open neighbourhoods of the identity in $G^\wedge$ and each of them is evidently a subgroup of $G^\wedge$. □

Proposition 2.10. *Let G be a precompact group of bounded order. Then G is reflexive if and only if it has the following two properties:*

(1) $G^\wedge$ *has no infinite compact subsets.*

(2) *Every compact subset of G is contained in a compact subgroup of G.*

Proof. If G is reflexive and $K \subset G^\wedge$ is compact, then K is finite by [1, Proposition 2.7]. By Lemma 2.9, $G^\wedge$ is protodiscrete. Hence (2) follows from Proposition 2.7.

Suppose conversely that (1) and (2) hold. Then, by Lemma 2.9, G satisfies the hypothesis of Proposition 2.7 and is therefore reflexive. $\square$

Combining Lemma 2.4 and Proposition 2.10 we deduce the following:

Corollary 2.11. *A pseudocompact group G of bounded order is reflexive if and only if each compact subset of G is contained in a compact subgroup of G.*

Remark 2.12. Proposition 2.10 is false for groups of infinite exponent. It suffices to consider an infinite torsion-free pseudocompact group G without infinite compact subsets (see Corollary 5.6 of [19]). Such a group is reflexive by (3) of Theorem 1.1, but it cannot contain any nontrivial compact subgroup.

3. Some extension results

We say that $\mathcal{P}$ is a *three space property* if for every Hausdorff topological group G and every closed invariant subgroup N of G such that both N and G/N have $\mathcal{P}$, the group G has $\mathcal{P}$ as well. It is known, on one hand, that compactness, connectedness, precompactness, pseudocompactness, Raĭkov completeness, etc., are all three space properties. On the other hand, Lindelöfness, normality, having a countable network, countable compactness, and many others, fail to be three space properties (see [8] for a detailed discussion on the subject).

We have already mentioned in the introduction that extensions of reflexive groups by compact groups preserve reflexivity (provided that extensions are *MAP groups*, i.e., their dual groups separate points). However, we will see in Section 4 that an extension of a compact group by a reflexive (even ω-bounded) group may fail to be reflexive. It is worth noting that extensions preserve the class of precompact topological groups [12, Theorem 6.3], and that all precompact groups are MAP groups.

Therefore reflexivity is not a three space property even among ω-bounded groups. The situation is completely different for the class of P-groups as we now set on to show. First we need two auxiliary facts.

Lemma 3.1. *Let H be a closed subgroup of a topological Abelian group G. If the groups H and G/H are protodiscrete, so is G.*

Proof. Take an arbitrary neighborhood U of the neutral element e in G and choose another neighborhood V of e with $V + V \subset U$. Since H is protodiscrete, there exists an open subgroup C of H with $C \subset H \cap V$. Let W be an open symmetric neighborhood of e in G such that $W \subset V$ and $(W + W + W) \cap H \subset C$.

Denote by π the quotient homomorphism of G onto G/H. Since G/H is protodiscrete, one can find an open subgroup K of G/H satisfying $K \subset \pi(W)$. We claim that $N = \pi^{-1}(K) \cap (W + C)$ is an open subgroup of G with $N \subset U$. The set N is open in G since K is open in G/H and W is open in G. It is also clear that $N \subset W + C \subset V + V \subset U$. Therefore, to finish the proof, it suffices to verify that N is a subgroup of G.

First we note that N is symmetric, and $C \subset N$ (observe that $C \subset H$). In fact, our definition of N implies that $N + C = N$. We then take arbitrary elements $x_1, x_2 \in N$. There exist $w_1, w_2 \in W$ and $c_1, c_2 \in C$ such that $x_i = w_i + c_i$ for $i = 1, 2$. Note that $\pi(N) = K \cap \pi(W + C) = K \cap \pi(W) = K$. Since K is a subgroup of G/H, we see that $\pi(x_1 + x_2) = \pi(x_1) + \pi(x_2) \in K$. Hence there exists $x_3 \in N$ such that $\pi(x_3) = \pi(x_1 + x_2)$ and we can find $h \in H$ such that $x_1 + x_2 = x_3 + h$. Choose $w_3 \in W$ and $c_3 \in C$ such that $x_3 = w_3 + c_3$. Then $w_1 + w_2 - w_3 = h - c_1 - c_2 + c_3$, so the element $h - c_1 - c_2 + c_3$ is in $(W + W + W) \cap H \subset C$. In its turn, this implies that $h \in C$. Therefore, $x_1 + x_2 = x_3 + h \in N + C = N$. This shows that N is a subgroup of G and finishes the proof. $\square$

Proposition 3.2. *Let G be a topological group and H be a closed subgroup of G. Suppose that both H and G/H are reflexive, protodiscrete, and contain no infinite compact subsets. If $K \subset G^\wedge$ is compact, then there is an open subgroup N of G with $K \subset N^\perp$.*

Proof. Let K be a compact subset of $G^\wedge$. Denote by j the natural embedding of H to G. We consider the exact sequences

$$ H \xrightarrow{\hspace{1cm} j \hspace{1cm}} G \xrightarrow{\hspace{1cm} \pi \hspace{1cm}} G/H $$

and

$$ (G/H)^\wedge \xrightarrow{\hspace{1cm} \widehat{\pi} \hspace{1cm}} G^\wedge \xrightarrow{\hspace{1cm} \widehat{j} \hspace{1cm}} H^\wedge. $$

Then $\widehat{j}(K)$ is a compact subset of $H^\wedge$. Since H is reflexive and protodiscrete, there is an open subgroup P of H such that $\widehat{j}(K) \subset P_H^\perp$, where $P_H^\perp$ is the annihilator of P in $H^\wedge$. This implies that $K \subset P_G^\perp$, with $P_G^\perp$ the annihilator of P in $G^\wedge$.

Let now $\widetilde{P}$ be an open subgroup of G with $\widetilde{P} \cap H = P$ (here we use the protodiscreteness of G provided by Lemma 3.1) and consider the sequence of mappings

$$ \widetilde{P} \xrightarrow{\hspace{1cm} \pi_P \hspace{1cm}} (\widetilde{P} + H)/H \xrightarrow{\hspace{1cm} \sigma \hspace{1cm}} \widetilde{P}/P. $$

Here π_P is the quotient homomorphism and σ is the obvious group isomorphism given by the third isomorphism theorem. Since $\widetilde{P}$ is open in G, σ is a topological isomorphism.

We have the following dual sequence:

$$ \left(\widetilde{P}/P\right)^\wedge \xrightarrow{\hspace{1cm} \widehat{\sigma} \hspace{1cm}} \left((\widetilde{P} + H)/H\right)^\wedge \xrightarrow{\hspace{1cm} \widehat{\pi}_P \hspace{1cm}} \widetilde{P}^\wedge \; . $$

As neither $\widetilde{P}$ nor $(\widetilde{P} + H)/H$ have infinite compact subsets, $\widehat{\pi}_P$ (and hence also $\widehat{\pi}_P \circ \widehat{\sigma}$) is a topological isomorphism onto $P^\perp$, the annihilator of P in $\widetilde{P}^\wedge$.

Finally, denote by $R \colon G^\wedge \to \widetilde{P}^\wedge$ the restriction homomorphism dual to the inclusion of $\widetilde{P}$ into G. Since $K \subset P_G^\perp$, $R(K)$ is contained in the image of $\widehat{\pi}_P$. Then $\widehat{\pi}_P^{-1}(R(K))$ is a compact subset of the dual of the group $(\widetilde{P} + H)/H$. Notice that $(\widetilde{P} + H)/H$ is reflexive as an open subgroup of the reflexive group G/H [6, Theorem 2.3]. Since the former group is protodiscrete, there is an open subgroup N of $\widetilde{P}$ (hence of G) such that $\widehat{\pi}_P^{-1}(R(K)) \subset \pi_P(N)^\perp$.

We finally claim that $K \subset N^\perp$. To that end, let $\chi \in K$ and $x \in N$ be arbitrary elements. Since $\widehat{j}(K) \subset P_H^\perp$, there is some $\widetilde{\chi} \in \left(\widetilde{P}/P\right)^\wedge$ with $R(\chi) = (\sigma \circ \pi_P)^\wedge(\widetilde{\chi})$. But

then $\widehat{\sigma}(\widetilde{\chi}) \in \widehat{\pi}_P^{-1}(R(K))$ and, recalling that $\widehat{\pi}_P^{-1}(R(K)) \subset \pi_P(N)^{\perp}$, we see that

$$\widehat{\sigma}(\widetilde{\chi})(\pi_P(x)) = 1. \tag{1}$$

In addition,

$$\widehat{\sigma}(\widetilde{\chi})(\pi_P(x)) = \widehat{\pi}_P(\widehat{\sigma}(\widetilde{\chi}))(x) = (\sigma \circ \pi_P)^{\wedge}(\widetilde{\chi})(x) = R(\chi)(x) = \chi(x). \tag{2}$$

Equalities (1) and (2) show that $\chi(x) = 1$ for all $x \in N$ and $\chi \in K$. Hence $K \subset N^{\perp}$. $\square$

Corollary 3.3. *Let H be a closed subgroup of a topological group G and assume that H and G/H contain no infinite compact subsets. If both H and G/H are reflexive and protodiscrete, then G is reflexive and protodiscrete.*

Proof. It follows from our assumptions about H and G/H that all compact subsets of the group G are finite, while Lemma 3.1 implies that G is protodiscrete and, therefore, locally quasi-convex.

Take a basic open neighborhood of the neutral element in $G^{\wedge\wedge}$ of the form $K^{\triangleright}$, where K is a compact subset of $G^{\wedge}$. By Proposition 3.2, there exists an open subgroup N of G such that $K \subset N^{\perp}$. Then $\alpha_G(N) \subset (N^{\perp})^{\triangleright} \subset K^{\triangleright}$, so α_G is continuous.

Since G is locally quasi-convex, α_G is necessarily injective and open as a mapping onto $\alpha_G(G)$ [3, Proposition 6.10]. Finally, $G^{\wedge}$ carries the topology of pointwise convergence on elements of G since G does not have infinite compact subsets and, as a consequence, α_G is surjective (apply [13, Theorem 1.3]). We conclude that G is reflexive. $\square$

In Corollary 3.5 below we show that the property of being a reflexive P-group is closed under extensions. Let us first establish that the class of P-groups behaves similarly:

Lemma 3.4. *Suppose that H is a closed subgroup of a topological (not necessarily Abelian) group G and that both H and G/H are P-groups. Then G is also a P-group.*

Proof. Let $\pi \colon G \to G/H$ be the quotient homomorphism. Denote by τ the topology of G and let $\tilde{\tau}$ be the P-modification of the topology τ. It is clear that $\tilde{\tau}$ is finer than τ and that $\widetilde{G} = (G, \tilde{\tau})$ is again a topological group. In particular, H is closed in $\widetilde{G}$. Since H is

a P-group, we see that $\tilde{\tau} \restriction H = \tau \restriction H$. Similarly, since G/H is a P-group, the quotient groups $\widetilde{G}/H$ and G/H carry the same topology, i.e., $\tilde{\tau}/H = \tau/H$. By [15, Lemma 1], this implies that $\tilde{\tau} = \tau$, so G is a P-group. $\qquad\square$

Corollary 3.5. *If H is a closed subgroup of a topological group G and both H and G/H are reflexive P-groups, then so is G.*

Proof. It follows from Lemma 3.4 that G is a P-group. Notice that P-groups are protodiscrete [2, 4.4.1 (a)] and have no infinite compact subsets [22, 4.K.2]. Hence G is reflexive by Corollary 3.3. $\qquad\square$

4. EXTENDING COMPACT GROUPS BY REFLEXIVE GROUPS

In this section we present several examples of non-reflexive extensions of compact groups by ω-bounded groups. Let us start with two lemmas.

Lemma 4.1. *Let $\phi\colon H_1 \to H_2$ be a continuous homomorphism of topological groups and $Gr(\phi) = \{(x, \phi(x)) : x \in H_1\}$ be the graph of ϕ considered as a subgroup of $H_1 \times H_2$. If L is a dense subgroup of H_2, then the subgroup $G = \mathrm{Gr}(\phi) + N$ of $H_1 \times H_2$ is an extension of H_1 by L, where $N = \{e\} \times L$ and e is the neutral element of H_1. If L is G_δ-dense in H_2, then G is G_δ-dense in $H_1 \times H_2$.*

Proof. It is easy to see that $N = G \cap (\{e\} \times H_2)$, so N is a closed subgroup of G topologically isomorphic to L. Since N is dense in $\{e\} \times H_2$, it follows from [23, Lemma 1.3] that the restriction to G of the projection $\pi\colon H_1 \times H_2 \to H_1$ to the first factor is an open homomorphism of G onto H_1. Therefore, $G/N \cong H_1$.

Finally, suppose that L is a G_δ-dense subgroup of H_2 and let $V_1 \times V_2$ be a non-empty G_δ-subset of $H_1 \times H_2$. Take $x \in V_1$; by the G_δ-density of L there must be $y \in L \cap (\phi(x)^{-1} + V_2)$. Then $(x, \phi(x) + y) \in G \cap (V_1 \times V_2)$ and hence G is G_δ-dense in $H_1 \times H_2$. $\qquad\square$

Lemma 4.2. *Let $P = P\mathbb{Z}(2)^{\mathfrak{c}}$ be the P-modification of the compact group $\mathbb{Z}(2)^{\mathfrak{c}}$. There is a topological monomorphism $j\colon P^{\wedge} \to \mathbb{Z}(2)^{2^{\mathfrak{c}}}$ such that $j(P^{\wedge}) \cap \Sigma\mathbb{Z}(2)^{2^{\mathfrak{c}}} = \{e\}$.*

Proof. Let A be a maximal independent subset of P. Denote by j_A the restriction homomorphism of $P^\wedge$ to $\mathbb{Z}(2)^A$, $j_A(\chi) = \chi \restriction A$ for each $\chi \in P^\wedge$. We claim that j_A is a topological isomorphism of $P^\wedge$ onto the dense subgroup $j_A(P^\wedge)$ of $\mathbb{Z}(2)^A$.

Indeed, j_A is a monomorphism since A generates the group P algebraically. Since the compact subsets of P are finite, $P^\wedge$ is a topological subgroup of $\mathbb{Z}(2)^P$. Hence the continuity of j_A follows from the continuity of the projection of $\mathbb{Z}(2)^P$ to $\mathbb{Z}(2)^A$. Let us show that j_A is open as a mapping of $P^\wedge$ onto the subgroup $j_A(P^\wedge)$ of $\mathbb{Z}(2)^A$. Given a neighborhood U of the neutral element in $P^\wedge$, we can find elements $x_1, \ldots, x_n$ in P such that

$$V = \{\chi \in P^\wedge : \chi(x_i) = 1 \text{ for each } i = 1, \ldots, n\} \subset U.$$

For every $i \leq n$, take elements $a_{i,1}, \ldots, a_{i,k_i} \in A$ such that $x_i = a_{i,1} \cdot \cdots \cdot a_{i,k_i}$ and let

$$B = \{a_{i,j} : 1 \leq i \leq n, \ 1 \leq j \leq k_i\}$$

and

$$O = \{x \in \mathbb{Z}(2)^A : x(b) = 1 \text{ for each } b \in B\}.$$

An easy verification shows that $j_A(V) \supset O \cap j_A(P^\wedge)$, which implies that j_A is open. Summing up, j_A is a topological monomorphism. The density of $j_A(P^\wedge)$ in $\mathbb{Z}(2)^A$ is evident. Note that $j_A(\chi) \in \Sigma\mathbb{Z}(2)^A$ if and only if $\chi(a) = 1$ for all but countably many $a \in A$.

For every $C \in [\mathfrak{c}]^{\leq \omega}$, let $N_C = \pi_C^{-1}(e_C)$ be the G_δ-subset of $\mathbb{Z}(2)^\mathfrak{c}$, where $\pi_C \colon \mathbb{Z}(2)^\mathfrak{c} \to \mathbb{Z}(2)^C$ is the projection and e_C is the neutral element of $\mathbb{Z}(2)^C$. Let Y_C denote a maximal independent subset of N_C. Observe that, for every $x \notin N_C$, $x \cdot Y_C$ is again independent.

For every $B \in [\mathfrak{c}]^{\leq \omega}$, choose an element $x_B \in \mathbb{Z}(2)^\mathfrak{c}$ which is supported precisely on B, that is, $x_B(\alpha) = -1$ iff $\alpha \in B$.

We then define, for all $B, C \in [\mathfrak{c}]^{\leq \omega}$ with $B \cap C \neq \emptyset$, the sets $X_{B,C} = x_B \cdot Y_C$. Since each $X_{B,C}$ is independent and has cardinality $2^\mathfrak{c}$, we may construct as in Lemma 4.4 of [10] a collection of sets $Z_{B,C} \subset X_{B,C}$, such that:

(1) $|Z_{B,C}| = 2^\mathfrak{c}$;

(2) $Z_{B,C} \cap Z_{B',C'} = \emptyset$ if $(B,C) \neq (B',C')$;

(3) $\bigcup_{B,C} Z_{B,C}$ is independent.

Let Z be a maximal independent subset of $\mathbb{Z}(2)^{\mathfrak{c}}$ containing the union in (3). Denote by $j = j_Z$ the topological monomorphism of $P^{\wedge}$ to $\mathbb{Z}(2)^Z$ corresponding to Z.

Suppose now that $\psi \in P^{\wedge}$, $\psi \neq \mathbf{1}$. Since ψ is continuous there must be $C \in [\mathfrak{c}]^{\leq \omega}$ such that $\psi(N_C) = \{1\}$. As $\psi \neq \mathbf{1}$ (and noting that $\Sigma \mathbb{Z}(2)^{\mathfrak{c}}$ is dense in P), there must also be $B \in [\mathfrak{c}]^{\leq \omega}$ with $\psi(x_B) = -1$; observe that necessarily $B \cap C \neq \emptyset$. It follows that $\psi(x_B \cdot N_C) = \{-1\}$ and $\psi(Z_{B,C}) = \{-1\}$, which means that $j(\psi) \notin \Sigma \mathbb{Z}(2)^Z$. This implies that $j(P^{\wedge}) \cap \Sigma \mathbb{Z}(2)^Z = \{e\}$. Note that $|Z| = 2^{\mathfrak{c}}$, so we complete the proof by identifying Z with $2^{\mathfrak{c}}$. $\qquad \square$

The non-reflexivity of the pseudocompact group G in the next example is obtained making use of Corollary 2.11.

Example 4.3. *There is a non-reflexive pseudocompact group G that arises as an extension of a compact group $G/L \cong \mathbb{Z}(2)^{2^{\mathfrak{c}}}$ by a closed, reflexive, ω-bounded subgroup L of G.*

Proof. Let P be the P-modification of the compact group $\mathbb{Z}(2)^{\mathfrak{c}}$. By Theorem 4.8 of [20], P is reflexive. Denote by L the character group of P. It is easy to see that $|L| = 2^{\mathfrak{c}}$. By Lemma 4.2, there is a topological monomorphism $j \colon L \to \mathbb{Z}(2)^{2^{\mathfrak{c}}}$ such that $j(L)$ is dense in $\mathbb{Z}(2)^{2^{\mathfrak{c}}}$ and $j(L) \cap \Sigma \mathbb{Z}(2)^{2^{\mathfrak{c}}} = \{e\}$. In the sequel we identify L with $j(L)$.

Let K be a compact subset of $\mathbb{Z}(2)^{2^{\mathfrak{c}}}$ such that $|K| = 2^{\mathfrak{c}}$ and $\langle K \rangle$ is dense in $\mathbb{Z}(2)^{2^{\mathfrak{c}}}$. For example, one can take $K = \{e\} \cup \{b_{\alpha} : \alpha < 2^{\mathfrak{c}}\}$, where e is the neutral element of $\mathbb{Z}(2)^{2^{\mathfrak{c}}}$ and $b_{\alpha}(\beta) = -1$ only if $\alpha = \beta$; $\alpha, \beta < 2^{\mathfrak{c}}$. Let also R_K be a subgroup of $\mathbb{Z}(2)^{2^{\mathfrak{c}}}$ such that $\mathbb{Z}(2)^{2^{\mathfrak{c}}} = \langle K \rangle \oplus R_K$ and define a (necessarily discontinuous) homomorphism $\phi \colon \mathbb{Z}(2)^{2^{\mathfrak{c}}} \to \mathbb{Z}(2)^{2^{\mathfrak{c}}}$ with the following properties:

(1) $\phi \restriction \langle K \rangle$ is the identity mapping;

(2) $\phi(r) \notin rL$ for every $r \in R_K$ with $r \neq e$.

To construct such a homomorphism we first observe that $|\mathbb{Z}(2)^{2^{\mathfrak{c}}}/\langle K \rangle| = 2^{2^{\mathfrak{c}}}$ and then consider a maximal independent subset $\{r_\alpha \colon \alpha < 2^{2^{\mathfrak{c}}}\}$ of R_K. It suffices to define ϕ satisfying (1) such that $\phi(r_\alpha) \notin \langle \bigcup\{r_\beta L \cup \phi(r_\gamma)L \colon \beta \le \alpha,\ \gamma < \alpha\} \rangle$, for each $\alpha < 2^{\mathfrak{c}}$.

We use the homomorphism ϕ to apply Lemma 4.1 and consider the subgroup $G = \mathrm{Gr}(\phi) \cdot N$ of $\Pi = \mathbb{Z}(2)^{2^{\mathfrak{c}}} \times \mathbb{Z}(2)^{2^{\mathfrak{c}}}$, where $N = \{e\} \times L$. By Lemma 4.1, G is a G_δ-dense subgroup of the compact group Π, so G is pseudocompact according to [14, Theorem 1.2].

Then $\widetilde{K} = \{(k,k) \colon k \in K\}$ is a compact subset of G. If $(a,b) \in G \cap cl_\Pi\langle \widetilde{K} \rangle$, then $a = b$ and $a = \phi(a)y$ with $y \in L$. Since $a = kr$ with $k \in \langle K \rangle$ and $r \in R_K$, we see $ry^{-1} = \phi(r)$. Therefore (2) implies that $r = e$. This proves the inclusion

$$cl_G\langle \widetilde{K} \rangle \subset \{(a,a) \colon a \in \langle K \rangle\},$$

while the inverse inclusion is evident. Thus $\langle \widetilde{K} \rangle$ is a closed subgroup of G. Since $\langle K \rangle$ is a proper dense subgroup of $\mathbb{Z}(2)^{2^{\mathfrak{c}}}$ it follows that $\langle \widetilde{K} \rangle$ is not compact and, hence, the group G is not reflexive by Corollary 2.11. $\qquad\square$

We now construct a larger family of non-reflexive extensions of compact (even metrizable) groups by ω-bounded groups. This requires several preliminary steps.

Given an abstract Abelian group G, we denote by $G^{\#}$ the underlying group G which carries the maximal precompact group topology [16]. This topology on G is called the *Bohr* topology of G. Notice that *every* homomorphism of $G^{\#}$ to $\mathbb{T}$ is continuous and that this property characterizes $G^{\#}$ among precompact groups. The following fact is a part of the duality folklore.

Lemma 4.4. *Let K be a compact Abelian topological group and $(K^\wedge, \tau_p(K))$ be the dual group of K with topology $\tau_p(K)$ of pointwise convergence on elements of K. Then $\tau_p(K)$ is the Bohr topology of the (abstract) group $K^\wedge$.*

Proof. By the classical Pontryagin–van Kampen duality theorem $K^\wedge$ is a discrete group and $\alpha_K \colon K \to K^{\wedge\wedge}$ is a topological isomorphism. Therefore, the Bohr topology of $K^\wedge$ is the precompact group topology $\tau_p(K)$ generated by K. $\qquad\square$

Definition 4.5. Let $\kappa \geq \omega$ be a cardinal number. We say that a subgroup $L \leq \mathbb{Z}(2)^{2^\kappa}$ satisfies condition (Sm) provided that for every $N \in [L]^{\leq \omega}$ there is a set $A_N \subset 2^\kappa$ with $|A_N| = 2^\kappa$ such that $\pi_{A_N}(N) = \{e_N\}$, where $\pi_{A_N} \colon \mathbb{Z}(2)^{2^\kappa} \to \mathbb{Z}(2)^{A_N}$ is the projection and e_N is the neutral element of $\mathbb{Z}(2)^{A_N}$.

Given a cardinal κ, we denote by $\mathbb{Z}(2)^{(\kappa)}$ the direct sum of κ copies of the group $\mathbb{Z}(2)$.

Lemma 4.6. *Let $\kappa \geq \omega$ be a cardinal, L be a dense pseudocompact subgroup of $\mathbb{Z}(2)^{2^\kappa}$ with $|L| \leq 2^\kappa$, and suppose that $L^\wedge$ is discrete and satisfies (Sm). Then there exists a pseudocompact group G containing L as a closed subgroup with G/L compact and such that*

$$G^\wedge = \left(\mathbb{Z}(2)^{(\kappa)}\right)^{\#} \times L^\wedge,$$

where $\left(\mathbb{Z}(2)^{(\kappa)}\right)^{\#}$ stands for the group $\mathbb{Z}(2)^{(\kappa)}$ equipped with its Bohr topology.

Proof. Let $\mathcal{S}$ be the family of countable subgroups of $\mathbb{Z}(2)^\kappa$. Consider the family

$$\mathcal{A} = \{(S, f, h) \colon S \in \mathcal{S}, \ f \in Hom(S, \mathbb{Z}(2)), \ h \in Hom(S, L)\}.$$

It is clear that $|\mathcal{A}| = 2^\kappa$. Let us fix an injective mapping $\rho \colon \mathcal{A} \to 2^\kappa$ such that $\rho(S, f, h) \in A_{h(S)}$ for every $(S, f, h) \in \mathcal{A}$; here $A_{h(S)}$ is as in Definition 4.5. The existence of such a mapping ρ follows from the equalities $|\mathcal{A}| = 2^\kappa$ and $|A_S| = 2^\kappa$ for $S \in \mathcal{S}$. For each $(S, f, h) \in \mathcal{A}$, we consider a homomorphism $\varphi_\alpha \colon \mathbb{Z}(2)^\kappa \to \mathbb{Z}(2)$ extending f, where $\alpha = \rho(S, f, h)$. Finally, for each $\alpha < 2^\kappa$ we define a homomorphism $\phi_\alpha \colon \mathbb{Z}(2)^\kappa \to \mathbb{Z}(2)$ by the following rule:

$$\phi_\alpha = \begin{cases} \varphi_\alpha, & \text{if } \alpha = \rho(S, f, h) \text{ for some } (S, f, h) \in \mathcal{A}; \\ \mathbf{1} \text{ (constant)}, & \text{if } \alpha \notin \rho(\mathcal{A}). \end{cases}$$

Let ϕ be the diagonal product of ϕ_α's, $\phi \colon \mathbb{Z}(2)^\kappa \to \mathbb{Z}(2)^{2^\kappa}$.

We define the group $G = \mathrm{Gr}(\phi) \cdot N$ for these ϕ and N as in Lemma 4.1. Then G is a dense subgroup of $\Pi = \mathbb{Z}(2)^\kappa \times \mathbb{Z}(2)^{2^\kappa}$. According to Lemma 4.1, G is G_δ-dense in Π, so G is pseudocompact by [14, Theorem 1.2]. Let π be the projection of the product Π to the first factor $\mathbb{Z}(2)^\kappa$.

We now proceed to identify the dual group of G. This task requires an analysis of the structure of compact subsets of G. We start with the following fact:

Claim 1. *If S is a countable subgroup of G and the restriction of π to S is one-to-one, then S is h-embedded in G.*

Indeed, let S be a countable subgroup of G such that $\pi{\restriction}_S$ is one-to-one. Then S is the graph of a homomorphism $g \colon D \to \mathbb{Z}(2)^{2^\kappa}$, where $D = \pi(S)$. It follows from the definition of G that $g = h \cdot \phi {\restriction} D$, where h is a homomorphism of D to L.

Let $p \colon S \to \mathbb{Z}(2)$ be an arbitrary homomorphism. For every $z \in S$, put $f(\pi(z)) = p(z)$. Then $f \colon D \to \mathbb{Z}(2)$ is a homomorphism and $(D, f, h) \in \mathcal{A}$. With $\alpha = \rho(D, f, h)$ we have $\phi_\alpha {\restriction}_D = \varphi_\alpha {\restriction}_D = f$ and $h(x)_\alpha = 1$ for each $x \in D$ since $\alpha \in A_{h(D)}$.

For every $\alpha < 2^\kappa$, let $\pi_\alpha \colon \mathbb{Z}(2)^{2^\kappa} \to \mathbb{Z}(2)$ be the projection of the product group $\mathbb{Z}(2)^{2^\kappa}$ to the αth factor $\mathbb{Z}(2)_{(\alpha)}$. We claim that the homomorphism p coincides with the restriction to S of the continuous homomorphism $\pi_\alpha \circ \varpi$, where $\varpi \colon \Pi \to \mathbb{Z}(2)^{2^\kappa}$ is the projection of Π to the second factor. Indeed, take an arbitrary element $z = (x, y) \in S$. Then $y = h(x) \cdot \phi(x)$, since S is the graph of the homomorphism $h \cdot \phi {\restriction} D$. We have, on one side, that

$$p(z) = f(\pi(z)) = f(x) = \varphi_\alpha(x). \tag{3}$$

On the other side, it follows from our choice of $\alpha < 2^\kappa$ and the definition of φ that

$$\pi_\alpha(\varpi(z)) = \pi_\alpha(y) \ = \ \pi_\alpha(h(x)) \cdot \pi_\alpha(\phi(x))$$

$$= \ h(x)_\alpha \cdot \varphi_\alpha(x) = \varphi_\alpha(x).$$

Comparing the above equality and equality (3), we infer that $p(z) = \pi_\alpha(\varpi(z))$ for each $z \in S$, which proves that $p = \pi_\alpha \circ \varpi {\restriction} S$.

Thus, every homomorphism $p \colon S \to \mathbb{Z}(2)$ extends to a continuous homomorphism of G to $\mathbb{Z}(2)$. This proves Claim 1.

Claim 2. *If S is a countable subgroup of G and the restriction of π to S is one-to-one, then $\pi(cl_G(S)) = \pi(S)$.*

Let g be an element of G such that $\pi(g) \notin \pi(S)$. It suffices to show that $g \notin cl_G(S)$. It follows from our choice of g that the restriction of π to the subgroup $T = S \cdot \langle g \rangle$ of G is also one-to-one. Hence, by Claim 1, T is h-embedded in G. Let φ be a homomorphism of T to $\mathbb{Z}(2)$ such that $\varphi(g) = -1$ and $\varphi(S) = \{1\}$. Then φ is continuous on T and, therefore, $g \notin cl_G(S)$.

Claim 3. *The projection $\pi(K)$ is finite, for every compact set $K \subset G$.*

Suppose for a contradiction that $\pi(K)$ is infinite, for a compact subset K of G. We can assume without loss of generality that $\pi(K)$ does not contain the neutral element e of $\mathbb{Z}(2)^\kappa$. Clearly $\pi(K)$ contains a countable infinite independent subset, say X. Choose a subset Y of K such that $\pi(Y) = X$ and the restriction of π to Y is one-to-one. Then Y is countable and independent in G and the restriction of π to the subgroup $S = \langle Y \rangle$ of G is one-to-one. Let $C = K \cap cl_G(S)$. Then C is a compact subset of G and Claim 2 implies that $\pi(C) \subset \pi(S)$. It also follows from $Y \subset K \cap S \subset C$ and $\pi(Y) = X$ that the compact sets C and $\pi(C)$ are infinite. Since $\pi(C)$ is countable (hence metrizable) and $X \subset \pi(C)$, there exists a sequence $\{x_n : n \in \omega\} \subset X$ converging to an element $x^* \in \pi(C)$, where $x^* \neq x_n$ for each $n \in \omega$. Notice that $x^* \neq e$. By induction we can choose an infinite subset X' of $\{x_n : n \in \omega\}$ such that $x^* \notin \langle X' \rangle$. Take a subset Y' of Y such that $\pi(Y') = X'$ and let $S' = \langle Y' \rangle$. Arguing as above, we see that $C' = K \cap cl_G(S')$ is a compact subset of G, $\pi(C') \subset \pi(S')$, and that $X' = \pi(Y') \subset \pi(C')$. In particular, the compact set $\pi(C')$ contains infinitely many points x_n's and, hence, $x^* \in \pi(C')$. The latter, however, is impossible since $\pi(C') \subset \pi(S') = \langle X' \rangle$ and $x^* \notin \langle X' \rangle$. This proves Claim 3.

We now obtain a complete description of $G^\wedge$, both algebraic and topological. Since G is dense in Π (and hence each character of G extends to a character of Π), there exists a natural (abstract) isomorphism

$$G^\wedge \cong (\mathbb{Z}(2)^\kappa \times \mathbb{Z}(2)^{2^\kappa})^\wedge \cong (\mathbb{Z}(2)^\kappa)^\wedge \oplus (\mathbb{Z}(2)^{2^\kappa})^\wedge. \tag{4}$$

The second isomorphism in (4) is obtained by restricting every character of Π to the factors $\mathbb{Z}(2)^{\kappa}$ and $\mathbb{Z}(2)^{2^{\kappa}}$. Since the groups $\mathbb{Z}(2)^{\kappa}$ and $\mathbb{Z}(2)^{2^{\kappa}}$ are compact, $(\mathbb{Z}(2)^{\kappa})^{\wedge}$ and $(\mathbb{Z}(2)^{2^{\kappa}})^{\wedge}$ are Boolean groups of cardinality κ and 2^{κ}, respectively.

Finally, we claim that $G^{\wedge}$ is topologically isomorphic to the group

$$((\mathbb{Z}(2)^{\kappa})^{\wedge}, \tau_p) \oplus L^{\wedge}, \tag{5}$$

where τ_p stands for the pointwise convergence topology on the abstract group $(\mathbb{Z}(2)^{\kappa})^{\wedge}$.

For each $F \subset \mathbb{Z}(2)^{\kappa}$ and each $P \subset G$, let

$$C_{F,P} = \{(x, \phi(x)y) : x \in F,\ y \in P\}.$$

If $K \subset G$ is compact, then $F = \pi(K)$ is finite and the set

$$P_K = \{y \in L : (x, \phi(x)y) \in K \text{ for some } x \in F\}$$

is a compact subset of L. Since $K \subset C_{F,P_K}$, we deduce that the family

$$\{(C_{F,P})^{\triangleright} : F \subset \mathbb{Z}(2)^{\kappa},\ |F| < \omega,\ P \subset L,\ P \text{ is compact}\} \tag{6}$$

forms a local base at the neutral element of the group $G^{\wedge}$.

Let now $F \subset \pi(G)$ be finite and $P \subset L$ be compact. If $\widetilde{P} = \phi(F)P$, then

$$F^{\triangleright} \times \widetilde{P}^{\triangleright} \subset (C_{F,P})^{\triangleright}.$$

The sets $(C_{F,P})^{\triangleright}$ are therefore neighborhoods of the identity in $((\mathbb{Z}(2)^{\kappa})^{\wedge}, \tau_p) \times L^{\wedge}$.

We now use the fact that $L^{\wedge}$ is discrete and take a compact set $P_0 \subset L$ such that $P_0^{\triangleright} = \{\mathbf{1}\}$, where $\mathbf{1}$ is the neutral element of $L^{\wedge}$. Since C_{F,P_0} contains the set $\{e\} \times P_0$, we see that $F^{\triangleright} \times \{e\} = (C_{F,P_0})^{\triangleright}$, for every finite set $F \subset \mathbb{Z}(2)^{\kappa}$. Therefore, $G^{\wedge}$ is topologically isomorphic to $((\mathbb{Z}(2)^{\kappa})^{\wedge}, \tau_p) \times L^{\wedge}$. It only remains to observe that, by Lemma 4.4, $((\mathbb{Z}(2)^{\kappa})^{\wedge}, \tau_p)$ is exactly $\left(\mathbb{Z}(2)^{(\kappa)}\right)^{\#}$. $\qquad\square$

Corollary 4.7. *There exists a pseudocompact, non-compact, Abelian group G which contains a closed ω-bounded (hence countably compact) subgroup N such that the dual groups*

$N^\wedge$ and $(G/N)^\wedge$ are discrete, but $G^\wedge$ is not. In addition, the quotient group G/N is compact and metrizable, while the bidual group $G^{\wedge\wedge}$ is compact and topologically isomorphic to ϱG, the completion of G.

Proof. In Lemma 4.6, let $\kappa = \omega$ and take L to be the Σ-product $\Sigma\Pi$, where $\Pi = \mathbb{Z}(2)^{2^{\mathfrak{c}}}$. It is obvious that L is ω-bounded, non-compact, and dense in Π. Also, it satisfies condition (Sm), and the dual group $L^\wedge$ is discrete. Applying Lemma 4.6, we find a dense pseudocompact subgroup G of $\mathbb{Z}(2)^\omega \times \Pi$ containing L as a closed subgroup such that $G/L \cong \mathbb{Z}(2)^\omega$ and $G^\wedge \cong (\mathbb{Z}(2)^{(\omega)})^\# \times L^\wedge$. In particular, G is not compact and $G^\wedge$ is not discrete. Further, the standard calculation shows that

$$G^{\wedge\wedge} = \mathbb{Z}(2)^\omega \times L^{\wedge\wedge} = \mathbb{Z}(2)^\omega \times \mathbb{Z}(2)^{2^{\mathfrak{c}}}.$$

Since G is dense in $\mathbb{Z}(2)^\omega \times \mathbb{Z}(2)^{2^{\mathfrak{c}}}$, we conclude that $G^{\wedge\wedge} \cong \varrho G$. $\square$

Remark 4.8. *The group G in Corollary 4.7 fails to be countably compact, even when the closed subgroup L of G is countably compact (even ω-bounded) and the quotient group $G/L \cong \mathbb{Z}(2)^\omega$ is compact and metrizable. The first example of a such a group was constructed in* [7].

Let us show that the present group G contains an infinite closed discrete subset. We keep the notation adopted in Lemma 4.6. Take a sequence $\{x_n : n \in \omega\} \subset \mathbb{Z}(2)^\omega$ converging to an element $x^* \in \mathbb{Z}(2)^\omega$, where $x^* \neq e$. We can assume that for each $n \in \omega$, the element x_n is not in the subgroup of $\mathbb{Z}(2)^\omega$ generated by the set $\{x^*\} \cup \{x_k : k < n\}$. For every $n \in \omega$, let $z_n = (x_n, y_n)$, where $y_n = \phi(x_n)$. It follows from the definition of G that the set $P = \{z_n : n \in \omega\}$ is contained in G.

We claim that P is closed and discrete in G. Since $\pi(z_n) = x_n$ and $x_n \to x^*$, all accumulation points of P, if any, lie in $\pi^{-1}(x^*)$. Take an arbitrary point $z = (x^*, y) \in \pi^{-1}(x^*) \cap G$. Again, our definition of G implies that $y = \phi(x^*) \cdot s$, for some $s \in L$. Denote by D the subgroup of $\mathbb{Z}(2)^\omega$ generated by $\{x^*\} \cup \{x_n : n \in \omega\}$ and take a homomorphism $f: D \to \mathbb{Z}(2)$ such that $f(x^*) = -1$ and $f(x_n) = 1$, for each $n \in \omega$. Since $\mathrm{supp}(s) \subset \mathfrak{c}$ is countable and the mapping $\varrho: \mathcal{A} \to \mathfrak{c}$ is injective, there exists a homomorphism $h: D \to L$

such that $\alpha = \varrho(D, f, h) \notin \mathrm{supp}(s)$. Notice that $\varphi_\alpha {\upharpoonright}_D = f$. We now have that

$$\pi_\alpha \varpi(z) = \pi_\alpha(y) \;\; = \;\; \pi_\alpha(\phi(x^*) \cdot s)$$

$$= \;\; \pi_\alpha \phi(x^*) \cdot \pi_\alpha(s) = \varphi_\alpha(x^*) = f(x^*) = -1,$$

while a similar calculation shows that $\pi_\alpha \varpi(z_n) = 1$, for each $n \in \omega$. Since the homomorphism $\pi_\alpha \circ \varpi \colon \mathbb{Z}(2)^\omega \times \mathbb{Z}(2)^{\mathfrak{c}} \to \mathbb{Z}(2)_{(\alpha)}$ is continuous, we conclude that $z \notin cl_G P$. This proves that P is closed and discrete in G and that G is not countably compact.

5. P-MODIFICATION OF REFLEXIVE GROUPS

In our first example we show that the P-modification of a reflexive σ-compact group can fail to be reflexive. Our argument uses essentially Pestov's theorem about the reflexivity of free Abelian topological groups on zero-dimensional compact spaces (see [28, 29]).

Let D be an uncountable discrete space and X a one-point compactification of D with a single non-isolated point x_0.

Lemma 5.1. *Let $G = A(X)$ be the free Abelian topological group over X. Then the P-modification PG of G is topologically isomorphic to the free Abelian topological group $A(Y)$, where $Y = PX$ is the P-modification of the space X.*

Proof. It is clear that Y is a Lindelöf P-space, and so is every finite power of Y [26]. Hence the group $A(Y)$ is Lindelöf, while [2, Proposition 7.4.7] implies that $A(Y)$ is a P-space.

Let $i \colon A(Y) \to A(X)$ be the continuous isomorphism of $A(Y)$ onto $A(X)$ which extends the identity mapping of Y onto X. It suffices to verify that i is a homeomorphism of $A(Y)$ onto $PA(X)$ (the group $A(X)$ with the P-modified topology).

Let C be a countable subset of D. Denote by r_C the retraction of X onto $X_C = C \cup \{x_0\}$, where $r_C(x) = x$ for each $x \in X_C$ and $r_C(y) = x_0$ for each $y \in X \setminus X_C$. Clearly, r_C is continuous. Extend r_C to a continuous homomorphism $R_C \colon A(X) \to A(X_C)$. Since the free Abelian topological group $A(X_C)$ has countable pseudocharacter (and R_C is continuous), $\ker R_C$ is a closed G_δ-set in $A(X)$. Hence $H_C = i^{-1}(\ker R_C)$ is an open subgroup of $A(Y)$.

Consider the family

$$\mathcal{H} = \{H_C : C \subset D, \ |C| \leq \omega\}.$$

It is easy to see that if $\{C_n : n \in \omega\}$ is a sequence of countable subsets of D and $C = \bigcup_{n \in \omega} C_n$, then $H_C \subset \bigcap_{n \in \omega} H_{C_n}$. It is also clear that the intersection of the family $\mathcal{H}$ contains only the neutral element of $A(Y)$.

We claim that $\mathcal{H}$ is a local base at the neutral element e of $A(Y)$. Indeed, take an arbitrary open neighborhood U of e in $A(Y)$. Suppose for a contradiction that every element of $\mathcal{H}$ meets the closed subset $F = A(Y) \setminus U$ of $A(Y)$. Then the family $\mathcal{H} \cup \{F\}$ of closed subsets of $A(Y)$ has the countable intersection property. Since the space $A(Y)$ is Lindelöf, we must have that $F \cap \bigcap \mathcal{H} \neq \emptyset$, which contradicts the equality $\bigcap \mathcal{H} = \{e\}$.

Finally, since $\ker R_C$ is open in $PA(X)$, the P-modification of $A(X)$, this implies that $i \colon A(Y) \to PA(X)$ is a homeomorphism. $\square$

The proof of the following fact is implicitly contained in (the proof of) Lemma 17.11 of [5]. We include the proof for the sake of completeness, it follows that the same result is valid for any infinite direct sum or any infinite Σ-product of compact groups (see [11, Lemma 3.12])

Lemma 5.2. *The dual group $\left(\Sigma \mathbb{T}^D\right)^{\wedge}$ of the Σ-product $\Sigma \mathbb{T}^D$ is a discrete group.*

Proof. Define for each $x \in \mathbb{T}$ and $d \in D$ the element of $f_{x,d} \in \mathbb{T}^D$ that takes the value x at d and 1 elsewhere. The set $K = \{f_{x,d} \colon x \in \mathbb{T}, \ d \in D\}$ is then a compact subset of $\Sigma \mathbb{T}^D$. Since for each $d \in D$, the set $A_d = \{f_{x,d} \colon x \in \mathbb{T}\}$ is a subgroup of $\mathbb{T}^D$ contained in K, we conclude that $\chi(A_d) = \{1\}$ for every $\chi \in K^{\triangleright}$ and $d \in D$. But the subgroup generated by all the A_d's is dense in $\mathbb{T}^D$, therefore $K^{\triangleright} = \{1\}$ and $\left(\Sigma \mathbb{T}^D\right)^{\wedge}$ is discrete. $\square$

Example 5.3. *The free Abelian topological group $G = A(X)$ is reflexive, but the P-modification PG of G is not. Furthermore, the second dual of PG is discrete.*

Proof. Since X is a zero-dimensional compact space, the reflexivity of G follows from [28] or [29]. Let us verify that PG is not reflexive. Since, by Lemma 5.1, $PA(X)$ is topologically

isomorphic to $A(Y)$, where $Y = PX$, and neither Y nor $A(Y)$ is discrete, it suffices to show that the second dual of $A(Y)$ is discrete.

Clearly, all functionally bounded subsets of the P-space Y are finite, so Y is a μ-space. It follows from [18, Theorem 2.1] that the dual group $A(Y)^\wedge$ is topologically isomorphic to $C_p(X, \mathbb{T})$, the group of all continuous functions on X with values in the circle group $\mathbb{T}$, endowed with the pointwise convergence topology.

Let $Y = D \cup \{x_0\}$, where x_0 is the only non-isolated point of Y. Every neighborhood of x_0 in Y has the form $Y \setminus C$, where C is a countable subset of D. Therefore, for every element $f \in C_p(Y, \mathbb{T})$, there exists a countable set $C \subset D$ such that $f(x) = f(x_0)$ for each $x \in Y \setminus C$.

Denote by $\Sigma \mathbb{T}^D$ the Σ-product lying in $\mathbb{T}^D$ and considered as a dense subgroup of the compact group $\mathbb{T}^D$. We consider a mapping $\varphi \colon C_p(Y, \mathbb{T}) \to \mathbb{T} \times \Sigma \mathbb{T}^D$ defined by $\varphi(f) = (f(x_0), t_f \cdot f)$, where $t_f = f(x_0)^{-1} \in \mathbb{T}$ and the function $t_f \cdot f$ is restricted to D. Then $t_f \cdot f \in \Sigma \mathbb{T}^D$ and $\varphi(f) \in \mathbb{T} \times \Sigma \mathbb{T}^D$. Since $C_p(Y, \mathbb{T})$ carries the topology of pointwise convergence, φ is a topological isomorphism of $C_p(Y, \mathbb{T})$ onto $\mathbb{T} \times \Sigma \mathbb{T}^D$. Hence the dual of $C_p(Y, \mathbb{T})$ is topologically isomorphic to $(\mathbb{T} \times \Sigma \mathbb{T}^D)^\wedge \cong \mathbb{T}^\wedge \times (\Sigma \mathbb{T}^D)^\wedge \cong \mathbb{Z}_d \times (\Sigma \mathbb{T}^D)^\wedge$, where $\mathbb{Z}_d$ is the discrete group of integers. Finally, we know by Lemma 5.2 that the dual group $(\Sigma \mathbb{T}^D)^\wedge$ is discrete. Hence the second dual $A(Y)^{\wedge\wedge}$ is discrete as well. $\square$

Corollary 5.4. *Let D be an uncountable discrete space and let Y denote the one-point Lindelöfication of D. Then $C_p(Y, \mathbb{T})$ is not reflexive.*

Proof. We note that $Y = PX$, where X is the one-point compactification of D. The proof of Example 5.3 shows that $C_p(Y, \mathbb{T})^\wedge$ is discrete, while $C_p(Y, \mathbb{T})$, being a proper dense subgroup of $\mathbb{T}^Y$, is not compact. $\square$

Remark 5.5. Corollary 5.4 is in contrast with Example 3.12 of [25], where it is shown that $C_p(Y, \mathbb{C})$ is reflexive if $|D| = \omega_1$ and $\mathrm{MA}(\omega_1)$ is assumed (here $\mathbb{C}$ stands for the field of complex numbers).

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Jorge Galindo, IMAC, Instituto Universitario de Matemáticas y Aplicaciones, Universidad Jaume I, E-12071, Castellón, Spain.

E-mail: `jgalindo@mat.uji.es`

M. Tkachenko, Depto. de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa, C.P. 09340, Mexico D.F., Mexico

E-mail: `mich@xanum.uam.mx`

M. Bruguera, Dept. de Matemática Aplicada I, Universidad Politécnica de Cataluña, Barcelona, Spain e-mail: m.montserrat.bruguera@upc.es

E-mail: `m.montserrat.bruguera@upc.es`

C. Hernández, Depto. de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa, C.P. 09340, Mexico D.F., Mexico

E-mail: `chg@xanum.uam.mx`