ISOMETRIES ON SPACES OF ABSOLUTELY CONTINUOUS FUNCTIONS IN A NONCOMPACT FRAMEWORK

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Abstract. In this paper we deal with surjective linear isometries between spaces of scaler-valued absolutely continuous functions on arbitrary (not necessarily closed or bounded) subsets of the real line (with at least two points). As a corollary, it is shown that when the underlying spaces are connected, each surjective linear isometry of these function spaces is a weighted composition operator, a result which generalizes all the previous known results concerning such isometries.

1. Introduction

The Banach-Stone theorem is a classical result in the theory of function spaces which describes all linear isometries from $C(X)$ onto $C(Y)$ as weighted composition operators based on a homeomorphism between the compact spaces X and Y. Stemming from this result, linear isometries on different contexts have been studied extensively. Indeed, the isometries of most of the well-known function spaces and algebras whose underlying spaces are (locally) compact have been described, similarly, as weighted composition operators (see, e.g., [4]). However, without assuming compactness, a linear isometry from $C_b(X)$ onto $C_b(Y)$ does not yield a homeomorphism between the Tychonoff spaces X and Y (see [5, 4M]), a fact which might explain the scarcity of results concerning isometries between function spaces in a noncompact framework (see [1] and [2]).

In this paper we study surjective linear isometries defined between spaces of scaler-valued absolutely continuous functions on arbitrary subsets of the real line (with at least two points). We use, following the direction of [7], a natural norm $\|\cdot\|$ in this context and show how $\|\cdot\|$-isometries are related to supremum norm isometries. It should be noted that we provide an example which shows that the space of absolutely continuous functions is not uniformly dense in the space of all bounded uniformly continuous functions and, consequently, the known results concerning supremum norm isometries cannot be used in this context. Indeed, we have to apply some technical lemmas to obtain the description of the isometries, which turns out to be based on a homeomorphism between

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the closure of the domains. As a consequence, we get generalizations of [7, Example 5] and [6, Corollary 4.4] to a noncompact framework.

2. Preliminaries

Let X be a subset of the real line $\mathbb{R}$ with at least two points. We recall that a scalar-valued function f on X has bounded variation if the total variation $\mathcal{V}(f)$ of f is finite, i.e.,

$$\mathcal{V}(f) := \sup\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : n \in \mathbb{N}, x_0, x_1, ..., x_n \in X, x_0 < x_1 < ... < x_n\} < \infty.$$

Moreover, a scalar-valued function f on X is said to be *absolutely continuous* if given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon,$$

for every finite family of non-overlapping open intervals $\{(a_i, b_i) : i = 1, \cdots, n\}$ whose extreme points belong to X with $\sum_{i=1}^{n}(b_i - a_i) < \delta$. We denote by $AC_b(X)$ the space of all scaler-valued absolutely continuous functions of bounded variation on X, equipped with the norm $\|\cdot\| = \max\{\|\cdot\|_\infty, \mathcal{V}(\cdot)\}$, where $\|\cdot\|_\infty$ denotes the supremum norm of a function. Let us remark that when X is bounded, each absolutely continuous function is automatically of bounded variation, and in this case we simply write $AC_b(X) = AC(X)$.

Given a scalar-valued function f on X, we denote the cozero set and the support of f by $coz(f)$ and $\mathrm{Supp}(f)$, respectively. For the case where f is bounded, we denote the maximum modulus set of f by $M_f = \{x \in X : |f(x)| = 1 = \|f\|_\infty\}$.

Meantime, for any $f \in AC_b(X)$, let $\widetilde{f}$ be the unique extension of f to the Stone-Čech compactification, βX, of X.

3. The results

From now on, we shall assume that X and Y are arbitrary (not necessarily closed or bounded) subsets of the real line with at least two points. Moreover, T will stand for a linear $\|\cdot\|$-isometry from $AC_b(X)$ onto $AC_b(Y)$ with respect to the norm $\|\cdot\|$ such that $T1$ is bounded away from zero, which is to say that there exists $t > 0$ such that, for each $y \in Y$, we have $|T1(y)| \geq t$. In particular, this is clearly the case when $T1$ is a unimodular function. Furthermore, it is shown that if the underlying spaces X and Y are connected, then $T1$ is bounded away from zero (see Corollary 3.15).

Note also that when the underlying spaces X and Y are compact, this condition that "$T1$ is bounded away from zero" coincides with property **P** in [2] and property **Q** in [3] (see also [6]).

Lemma 3.1. *Each absolutely continuous function f on X has a unique absolutely continuous extension $\overline{f}$ to the closure $\bar{X}$ of X.*

Proof. Since f is uniformly continuous, f has a unique uniformly continuous extension to the closure $\bar{X}$, which we denote by $\overline{f}$. We claim that $\overline{f}$ is absolutely continuous. To this end, let $\epsilon > 0$ and choose $\delta > 0$ associated to the absolutely continuity of f with respect to $\frac{\epsilon}{3}$. Assume that $\{(a_i, b_i) : i = 1, \cdots, n\}$ is a finite family of non-overlapping open intervals whose extreme points belong to $\bar{X}$ and $\sum_{i=1}^{n}(b_i - a_i) < \frac{\delta}{3}$. With no loss of generality, assume that

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n.$$

Put

$$x_1 = a_1,\ x_2 = b_1,\ x_3 = a_2,\ \cdots, x_{2n-1} = a_n,\ x_{2n} = b_n.$$

For each $i \in \{1, \cdots, 2n\}$, consider $x'_i = x_i$ if $x_i \in X$, otherwise, if x_i does not belong to X, we choose x'_i in X as follows:

If $x_1 \notin X$, select $x'_1 \in X$ such that $|x_1 - x'_1| < \frac{\delta}{3n}$, $|\overline{f}(x_1) - \overline{f}(x'_1)| < \frac{\epsilon}{3n}$, and we have either $x'_1 < x_1$, or $x_1 < x'_1 < x_2$. If $x_2 \notin X$, choose $x'_2 \in X$ such that $|x_2 - x'_2| < \frac{\delta}{3n}$, $|\overline{f}(x_2) - \overline{f}(x'_2)| < \frac{\epsilon}{3n}$, and we have either

$$\max\{x'_1, x_1\} < x'_2 < x_2,\ \text{or } \mathrm{x_2 < x'_2 < x_3}.$$

By continuing this process, for $2 \leq i \leq 2n-1$, if $x_i \notin X$, take $x'_i \in X$ such that $|x_i - x'_i| < \frac{\delta}{3n}$, $|\overline{f}(x_i) - \overline{f}(x'_i)| < \frac{\epsilon}{3n}$, and we have either

$$\max\{x'_{i-1}, x_{i-1}\} < x'_i < x_i,\ \text{ or } \mathrm{x_i < x'_i < x_{i+1}}.$$

Meantime, for $i = 2n$, if $x_{2n} \notin X$, we choose $x'_{2n} \in X$ such that $|x_{2n} - x'_{2n}| < \frac{\delta}{3n}$, $|\overline{f}(x_{2n}) - \overline{f}(x'_{2n})| < \frac{\epsilon}{3n}$, and also $x_{2n} < x'_{2n}$ or $\max\{x'_{2n-1}, x_{2n}\} < x'_{2n} < x_{2n}$.

We rename again x'_i by a'_i if i is odd, and by b'_i if i is even. Hence we get $a'_1, \cdots, a'_n, b'_1, \cdots, b'_n \in X$ and $\{(a'_i, b'_i) : i = 1, ..., n\}$ is a finite family of non-overlapping open intervals whose extreme points belong to X. Also

$$\sum_{i=1}^{n}(b'_i - a'_i) \leq \sum_{i=1}^{n}(|b'_i - b_i| + |b_i - a_i| + |a_i - a'_i|) < \sum_{i=1}^{n}\frac{\delta}{3n} + \frac{\delta}{3} + \sum_{i=1}^{n}\frac{\delta}{3n} = \delta.$$

Thus it follows that

$$\begin{aligned}\sum_{i=1}^{n}|\overline{f}(b_i) - \overline{f}(a_i)| &\leq \sum_{i=1}^{n}(|\overline{f}(b_i) - \overline{f}(b'_i)| + |\overline{f}(b'_i) - \overline{f}(a'_i)| + |\overline{f}(a'_i) - \overline{f}(a_i)|) \\ &< \sum_{i=1}^{n}\frac{\epsilon}{3n} + \frac{\epsilon}{3} + \sum_{i=1}^{n}\frac{\epsilon}{3n} = \epsilon,\end{aligned}$$

which implies that $\overline{f}$ is absolutely continuous. □

As a consequence of this lemma, the spaces of absolutely continuous functions defined on an arbitrary subset of the real line and on its completion coincide. In the next lemmas, we shall assume that X and Y are closed subsets of the real line.

Lemma 3.2. *If $f \in AC_b(X)$ and $\|Tf\|_\infty > \mathcal{V}(Tf)$, then $\mathcal{V}(f) \leq \|f\|_\infty$.*

Proof. Let $f \in AC_b(X)$ and $y_0 \in \beta Y$ such that $|\widetilde{Tf}(y_0)| = \|Tf\|_\infty > \mathcal{V}(Tf)$. Suppose, contrary to what we claim, that $\mathcal{V}(f) > \|f\|_\infty$. Let ϵ be a positive scalar such that $\|f\|_\infty + \epsilon < \mathcal{V}(f)$.

As $y_0 \in \beta Y$, choose a net $(y_i)_i$ in Y such that $y_i \longrightarrow y_0$. Since $T1$ is bounded away from zero, there exists $t > 0$ such that for every i we have $|T1(y_i)| \geq t$. Then $|\widetilde{T1}(y_0)| \geq t$ because $\widetilde{T1}$ is a continuous function.

Meantime, since $\|f\|_\infty + \epsilon < \mathcal{V}(f)$, it is clear that

$$\begin{aligned}\|f \pm \epsilon\| &= \max\{\|f \pm \epsilon\|_\infty, \mathcal{V}(f \pm \epsilon)\} \\ &= \max\{\|f \pm \epsilon\|_\infty, \mathcal{V}(f)\} \\ &= \mathcal{V}(f) = \|f\|.\end{aligned}$$

On the other hand, we have $\|Tf \pm \epsilon T1\| = \|f \pm \epsilon\|$ and $\|Tf\| = \|f\|$. Now it easily follows that $\|Tf \pm \epsilon T1\| = \|Tf\| = |\widetilde{Tf}(y_0)|$, and so

$$|\widetilde{Tf}(y_0) \pm \epsilon \widetilde{T1}(y_0)| \leq \|Tf \pm \epsilon T1\|_\infty \leq \|Tf \pm \epsilon T1\| = |\widetilde{Tf}(y_0)|.$$

Then $|\widetilde{Tf}(y_0) \pm \epsilon \widetilde{T1}(y_0)| \leq |\widetilde{Tf}(y_0)|$ which implies that $\widetilde{T1}(y_0) = 0$. This contradicts the fact $|\widetilde{T1}(y_0)| \geq t$. Therefore, $\mathcal{V}(f) \leq \|f\|_\infty$. □

Lemma 3.3. *If $f \in AC_b(X)$, then $\|Tf\|_\infty = \|f\|_\infty$.*

Proof. We divide the proof of this lemma into three parts as follows:

(i) First we show that for any $f \in AC_b(X)$, $\|Tf\|_\infty \leq \|f\|_\infty$. We verify this part by an argument similar to the proof of [8, Proposition 1.3]. Let $f \in AC_b(X)$ and $y_0 \in \beta Y$ with $|\widetilde{Tf}(y_0)| = \|Tf\|_\infty$. Assume, on the contrary, that $\|f\|_\infty < \|Tf\|_\infty$. Let ϵ be a positive scalar such that $\|f\|_\infty + \epsilon < |\widetilde{Tf}(y_0)|$. Choose $\lambda > 0$ large enough so that $(\lambda + 1)|\widetilde{Tf}(y_0)| = \|\lambda \widetilde{Tf}(y_0) + Tf\|_\infty > \mathcal{V}(\lambda \widetilde{Tf}(y_0) + Tf) = \mathcal{V}(Tf)$. Then, taking into account Lemma 3.2, we have

$$\|\lambda T^{-1}(\widetilde{Tf}(y_0)) + f\|_\infty \geq \mathcal{V}(\lambda T^{-1}(\widetilde{Tf}(y_0)) + f).$$

Hence, from the above relations it follows that

$$\begin{aligned}\|\lambda T^{-1}(\widetilde{Tf}(y_0)) + f\|_\infty &\leq \|\lambda T^{-1}(\widetilde{Tf}(y_0))\|_\infty + \|f\|_\infty \\ &\leq \lambda \|T^{-1}(\widetilde{Tf}(y_0))\| + \|f\|_\infty = \lambda|\widetilde{Tf}(y_0)| + \|f\|_\infty \\ &< \lambda|\widetilde{Tf}(y_0)| + |\widetilde{Tf}(y_0)| - \epsilon = (\lambda + 1)|\widetilde{Tf}(y_0)| - \epsilon \\ &= \|\lambda \widetilde{Tf}(y_0) + Tf\| - \epsilon \\ &= \|\lambda T^{-1}(\widetilde{Tf}(y_0)) + f\| - \epsilon \\ &= \|\lambda T^{-1}(\widetilde{Tf}(y_0)) + f\|_\infty - \epsilon,\end{aligned}$$

which is a contradiction showing that $\|Tf\|_\infty \leq \|f\|_\infty$.

(ii) We claim that for each $x \in X$, $|T^{-1}1(x)| = 1$. Suppose, contrary to what we claim, that there exists $x_0 \in X$ and $|T^{-1}1(x_0)| < 1$. Note that $\|T^{-1}1\|_\infty = 1$, because from the above part we have

$$1 = \|1\|_\infty \leq \|T^{-1}1\|_\infty \leq \|T^{-1}1\| = \|1\| = 1.$$

Define the function h by $h(x) := 1 - |T^{-1}1(x)|$ for all $x \in X$. It is easy to see that $h \in AC_b(X)$. Moreover, $|h(x)| + |T^{-1}1(x)| = 1$ for all $x \in X$, $h(x_0) = 1 - |T^{-1}1(x_0)|$ and $Th = T1 - T(|T^{-1}1|)$. Since $Th \neq 0$, we have $1 < \max\{\|1 + Th\|_\infty, \|1 - Th\|_\infty\}$. On the other hand, again from (i) it follows that

$$\|1 \pm Th\|_\infty = \|T(T^{-1}1 \pm h)\|_\infty \leq \|T^{-1}1 \pm h\|_\infty.$$

Thus there exists $x' \in \beta X$ with $1 < \max\{|\widetilde{h}(x') + \widetilde{T^{-1}1}(x')|, |\widetilde{h}(x') - \widetilde{T^{-1}1}(x')|\}$. Consequently, $1 < |\widetilde{h}(x')| + |\widetilde{T^{-1}1}(x')| = 1$, which is a contradiction. Hence the claim has been proved.

(iii) Finally, let $f \in AC_b(X)$. By (i), $\|Tf\|_\infty \leq \|f\|_\infty$. Next, taking into account (ii), an assertion similar to the part (i) for T^{-1} shows that $\|f\|_\infty = \|T^{-1}(Tf)\|_\infty \leq \|Tf\|_\infty$. Therefore, $\|f\|_\infty = \|Tf\|_\infty$, as desired. □

Remark 3.4. From Lemma 3.3, one might think that all the results concerning $\|\cdot\|$-isometries on $AC_b(X)$-spaces could be deduced from similar ones concerning supremum norm isometries (see basically [1]) provided $AC_b(X)$ was uniformly dense in the space of all bounded (uniformly) continuous functions on X. However, such density result is not true as the following example shows: let $X = \mathbb{N}$, M be the set of odd numbers, N be the set of even numbers, and define $g(x) = 1$ if $x \in M$, and $g(x) = 0$ if $x \in N$. Then g is a bounded uniformly continuous function but there is no function f of bounded variation with $\|f - g\|_\infty < \frac{1}{3}$.

Lemma 3.5. *$T1$ is a unimodular constant function.*

Proof. If $|X| = 2$, then it is easily seen that $|Y| = 2$ and so the result follows from [6]. Otherwise, we can assume y_1, y_2, y_3 are distinct points in Y such that $y_1 < y_2 < y_3$. Define

$$f(y) = \left(\frac{y - y_1}{y_2 - y_1}\chi_{[y_1,y_2]}(y) + \frac{y - y_3}{y_2 - y_3}\chi_{(y_2,y_3]}(y)\right) \quad (y \in Y).$$

Clearly, $f \in AC_b(Y)$. Since $\|f\| = \mathcal{V}(f) = 2 > \|f\|_\infty = 1$ and T is an isometry with respect to $\|\cdot\|$ and $\|\cdot\|_\infty$, we get $\|T^{-1}f\| = 2 > \|T^{-1}f\|_\infty = 1$. Hence $\mathcal{V}(f \pm T\frac{1}{2}) = \|f \pm T\frac{1}{2}\| = \|T^{-1}f \pm \frac{1}{2}\| =$

$\mathcal{V}(T^{-1}f) = 2$. So it follows that

$$\begin{aligned} 2 = \mathcal{V}\left(f \pm T\frac{1}{2}\right) &\geq \left|\left(f \pm T\frac{1}{2}\right)(y_1) - \left(f \pm T\frac{1}{2}\right)(y_2)\right| \\ &+ \left|\left(f \pm T\frac{1}{2}\right)(y_2) - \left(f \pm T\frac{1}{2}\right)(y_3)\right| \\ &= \left|\pm\frac{1}{2}T1(y_1) - \left(1 \pm \frac{1}{2}T1(y_2)\right)\right| + \left|1 \pm \frac{1}{2}T1(y_2) - \left(\pm\frac{1}{2}T1(y_3)\right)\right| \\ &\geq \left|2 \pm \left(T1(y_2) - \left(\frac{T1(y_1)}{2} + \frac{T1(y_3)}{2}\right)\right)\right|, \end{aligned}$$

which implies that $T1(y_2) - (\frac{T1(y_1)}{2} + \frac{T1(y_3)}{2}) = 0$. Hence $T1(y_2) = \frac{T1(y_1)+T1(y_3)}{2}$. Using an argument similar to the part (ii) in the proof of Lemma 3.3, one can observe that $|T1(y_2)| = 1$. Now, from the fact that each point in the unit circle is an extreme point of the closed unit ball of $\mathbb{C}$, it follows that $T1(y_1) = T1(y_2) = T1(y_3)$. This argument shows $T1$ is a unimodular constant function. □

In the sequel, without loss of generality, we shall assume that T is unital, i.e., $T1 = 1$.

The next result may be considered as a version of the additive Bishop's lemma for absolutely continuous function spaces.

Lemma 3.6. *(1) Let $f \in AC_b(X)$ and $x_0 \in X$. If $f(x_0) = 0$, then for any $r > \|f\|_\infty$, there exists $h \in AC_b(X)$ such that $h(x_0) = 1$, $M_h = \{x_0\}$ and $\||f| + rh\|_\infty = \|f \pm rh\|_\infty = r$.*

(2) Assume that $f \in AC_b(X)$, $x_0 \in X$, $f(x_0) \neq 0$ and $r \geq \frac{\|f\|_\infty}{|f(x_0)|}$. Then there exists a non-negative function $u \in AC_b(X)$ such that $u(x_0) = 1$, $M_u = \{x_0\}$ and $\||f| + ru|f(x_0)|\|_\infty = \|f + ruf(x_0)\|_\infty = |f(x_0)|(1 + r)$.

Furthermore, for every scalar e with $|e| \geq |f(x_0)|$ we have $\||f| + ru|e|\|_\infty = |f(x_0)| + r|e|$.

Proof. (1) We prove this first part following the ideas given in the proof of [9, Lemma 1]. Assume that $f(x_0) = 0$ and $r > \|f\|_\infty$. Let $\{V_n\}$ be a decreasing sequence of neighborhoods of x_0 in X such that each $\overline{V_n}$ is compact and $\bigcap_{n=1}^\infty V_n = \{x_0\}$. Define

$$U_n = \left\{x \in V_n : |f(x)| < \frac{r - \|f\|_\infty}{2^{n+1}}\right\} \quad (n \in \mathbb{N}).$$

It is apparent that for each $n \in \mathbb{N}$, U_n is a neighborhood of x_0 in X, $U_{n+1} \subseteq U_n$ and $\bigcap_{n=1}^\infty U_n = \{x_0\}$. For any $n \in \mathbb{N}$, choose a function $h_n \in AC_b(X)$, $h_n(x_0) = 1$, $0 \leq h_n \leq 1$, $\mathcal{V}(h_n) \leq 2$, and $h_n = 0$ on $X \setminus U_n$. Put $h = r\sum_{n=1}^\infty \frac{h_n}{2^n}$. First we note that since $\|h\| \leq r\sum_{n=1}^\infty \frac{\|h_n\|}{2^n} \leq 2r$ and h has a compact support ($\mathrm{Supp}(h) \subseteq \overline{U_1} \subseteq \overline{V_1}$), the function h belongs to $AC_b(X)$. Clearly, $0 \leq h \leq 1$ and $h(x_0) = 1$. Finally, by an argument similar to [9], it can be checked that $\||f| + rh\|_\infty = \|f \pm rhe\|_\infty = r$.

(2) We prove the second part by an argument similar to the one in the proof of [6, Lemma 3.8]. Clearly, there is a decreasing sequence $\{V_n\}$ of neighborhoods of x_0 in X such that each $\overline{V_n}$ is

compact and $\bigcap_{n=1}^{\infty} V_n = \{x_0\}$. Put $e_0 = f(x_0)$. For any $n \in \mathbb{N}$, we define

$$U_n = \left\{ x \in V_n : ||f(x)| - |e_0|| < \frac{|e_0|}{2^{n+1}} \right\}.$$

It is obvious that U_n is a neighborhood of x_0 in X and $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose $u_n \in AC_b(X)$ such that $0 \leq u_n \leq 1$, $u_n(x_0) = 1$, $\mathcal{V}(u_n) \leq 2$, and $u_n = 0$ on $X \setminus \overline{V_n}$. Now, set $u = \sum_{n=1}^{\infty} \frac{u_n}{2^n}$. Since u has a compact support ($\mathrm{Supp}(u) \subseteq \overline{U_1} \subseteq \overline{V_1}$) and $\sum_{n=1}^{\infty} \frac{\|u_n\|}{2^n} \leq 2$, u belongs to $AC_b(X)$. By arguments similar to [6], one may observe $\||f|+ru|e_0|\|_\infty = \|f+rue_0\|_\infty = |e_0|(1+r)$, and that for every scalar e with $|e| \geq |e_0|$ we have $\||f| + ru|e|\|_\infty = |e_0| + r|e|$. □

Lemma 3.7. *T and T^{-1} are disjointness preserving maps, i.e., they map functions with disjoint cozeros to functions with disjoint cozeros.*

Proof. Taking into account Lemma 3.6 (2), the result can be obtained by an approach similar to [2, Proposition 4.7] and [6, Lemma 3.9]. □

Given $x \in X$, we define

$$\mathcal{F}_x = \{f \in AC_b(X) : f(x) = 1 = \|f\|_\infty\},$$

which is a non-empty set. We also set

$$\mathcal{I}_x := \bigcap \{M_{\widetilde{Tf}} : f \in \mathcal{F}_x\},$$

where $M_{\widetilde{Tf}} = \{y \in \beta Y : |\widetilde{Tf}(y)| = 1 = \|\widetilde{Tf}\|_\infty\}$. Let us also recall that $\widetilde{Tf}$ denotes the unique extension of Tf to the Stone-Čech compactification, βY, of Y.

Lemma 3.8. *Given $x \in X$, the set $\mathcal{I}_x$ is non-empty.*

Proof. It is a typical result in the context of supremum norm isometries, but we include its proof for the sake of completeness. Since βY is compact, it is enough to show that the family $\{M_{\widetilde{Tf}} : f \in \mathcal{F}_x\}$ has the finite intersection property. To see this, let $f_1, ..., f_n$ in $\mathcal{F}_x$. Define $f = \sum_{i=1}^{n} \frac{f_i}{n}$. It is clear that $f \in \mathcal{F}_x$. By Lemma 3.3, $\|\widetilde{Tf}\|_\infty = \|Tf\|_\infty = \|f\|_\infty = 1$, then there exists a point y in the compact set βY such that $|\widetilde{Tf}(y)| = 1$. Hence we have

$$1 = |\widetilde{Tf}(y)| = \left| \sum_{i=1}^{n} \frac{\widetilde{Tf_i}(y)}{n} \right| \leq \sum_{i=1}^{n} \frac{|\widetilde{Tf_i}(y)|}{n} \leq \sum_{i=1}^{n} \frac{\|\widetilde{Tf_i}\|_\infty}{n} = 1,$$

which yields that $|\widetilde{Tf_i}(y)| = 1$ for $i = 1, ..., n$. Thus $y \in \bigcap_{i=1}^{n} M_{\widetilde{Tf_i}}$. Therefore, we get $\mathcal{I}_x \neq \emptyset$, as desired. □

In the next lemma we show that the subset $\mathcal{I}_x$ of βY is indeed a subset of Y. To this end, let us first introduce two types of functions in $AC_b(X)$ as follows:

Type 1. There are $a, b \in \mathbb{R}$ such that $a < b$,

$$f(x) = \chi_{[b,+\infty)}(x) + \frac{x-a}{b-a}\chi_{[a,+\infty)} \quad (x \in X),$$

and $\{0,1\} \subseteq f(X)$.

Type 2. There are $a, b \in \mathbb{R}$ such that $a < b$,

$$f(x) = \chi_{(-\infty,a]}(x) + \frac{x-b}{a-b}\chi_{(a,b)} \quad (x \in X),$$

and $\{0,1\} \subseteq f(X)$.

Let also S_i denote the set of all functions of type i $(i = 1, 2)$.

Lemma 3.9. *Given $x \in X$, $\mathcal{I}_x$ is a subset of Y.*

Proof. If Y is compact, then the claim clearly holds. Otherwise, taking into account the closedness of X and Y, we are in one of the following cases:

Case 1. X and Y are unbounded both from below and from above. We first prove the following claim.

Claim 1. For each $f \in S_1 \cup S_2$, $\{0,1\} \subseteq Tf(Y) \subseteq [0,1]$.

Let $f \in S_1$. It is easy to find nonzero functions $g, h \in AC_b(X)$ such that $coz(f) \cap coz(g) = \emptyset$ and $coz(1-f) \cap coz(h) = \emptyset$. Hence $coz(Tf) \cap coz(Tg) = \emptyset$ and $coz(T(1-f)) \cap coz(Th) = \emptyset$ by Lemma 3.7. Now, since $Tg \neq 0$ and $Th \neq 0$, we conclude that there exist $y, y' \in Y$ such that $Tf(y) = 0$ and $T(1-f)(y') = 0$. Thus, from $T1 = 1$ it follows that $Tf(y') = 1$. Therefore, $\{0,1\} \subseteq Tf(Y)$. Finally, if $t \in Tf(Y)$ and $t \notin [0,1]$, then taking into account that $|t| + |1-t| > 1$ we conclude that $\mathcal{V}(Tf) > 1$, which is impossible because $\|Tf\| = 1 = \|Tf\|_\infty$. A similar discussion shows that the result holds for any function in S_2. Now, the proof of Claim 1 is completed.

Given $f \in S_1 \cup S_2$, we have Tf is continuous and $\mathcal{V}(f) \leq 1$. Hence, thanks to Claim 1, it is not difficult to check that there exists $y_0 \in \mathbb{R}$ such that we have one of the following forms

$$Tf|_{(-\infty,y_0]\cap Y} = 0 \quad \textit{and} \quad Tf|_{(y_0,+\infty)\cap Y} \neq 0,$$

or

$$Tf|_{(-\infty,y_0)\cap Y} \neq 0 \quad \textit{and} \quad Tf|_{[y_0,+\infty)\cap Y} = 0.$$

For each $x \in X$, define

$$f_x(z) = \frac{z-a}{x-a}\chi_{(a,x)}(z) + \frac{z-b}{x-b}\chi_{[x,b)}(z) \quad (z \in X),$$

for some $a, b \in \mathbb{R}$ with $a < x < b$.

We can find $f_1 \in S_1$ and $f_2 \in S_2$ such that $coz(f_1) \cap coz(f_2) = \emptyset$ and $coz(f_i) \cap coz(f_x) = \emptyset$ $(i = 1, 2)$. From the argument after Claim 1, it follows that $coz(Tf_x)$ is included in a bounded subset of $\mathbb{R}$. Thus $\mathrm{Supp}(Tf_x)$ is a compact subset of Y.

Now, since $f_x \in \mathcal{F}_x$ and $\mathrm{Supp}(Tf_x)$ is a compact subset of Y, one easily conclude that $\mathcal{I}_x \subseteq \mathrm{Supp(Tf_x)} \subseteq \mathrm{Y}$.

Case 2. X and Y are bounded below and unbounded above. In this case we first prove the following claim.

Claim 2. For each $f \in S_1$, we have $0 \in \overline{Tf(Y)}$ and $1 \in Tf(Y) \subseteq [0, 1]$. Moreover, for each $f \in S_2$, we have $1 \in \overline{Tf(Y)}$ and $0 \in Tf(Y) \subseteq [0, 1]$.

Let $f \in S_2$. We have

$$\|f + 1\| = \max\{2, 1\} = 2 = \|f\| + 1,$$

which, taking into account that $T1 = 1$, yields that $\|Tf + 1\| = 2 = \|Tf\| + 1$. Hence $\max\{\|Tf + 1\|_\infty, \mathcal{V}(Tf)\|\} = \max\{\|Tf\|_\infty, \mathcal{V}(Tf)\|\} + 1$, which yields $\|Tf\|_\infty + 1 = \|Tf + 1\|_\infty$ because $\|Tf\| = 1 = \|Tf\|_\infty$. Thus there is a sequence $\{y_n\}$ in Y such that $\{Tf(y_n)\}$ is convergent and $|Tf(y_n) + 1| \longrightarrow 2$. Now it is easily derived that $Tf(y_n) \longrightarrow 1$. Therefore, $1 \in \overline{Tf(Y)}$.

Now, choose $h \in AC_b(X)$ such that $coz(f) \cap coz(h) = \emptyset$. Then $coz(Tf) \cap coz(Th) = \emptyset$ because T is a disjointness preserving map by Lemma 3.7, and as a consequence, since $Th \neq 0$, we have $0 \in Tf(Y)$. (Note that there is not necessarily such a function for $1 - f$ (compare with Claim 1). For example, let $X = [0, +\infty)$ and define $f(x) = (-x + 1)\chi_{[0,1]}(x)$.) Similarly to Case 1, it can be checked that $Tf(Y) \subseteq [0, 1]$.

Now take $f \in S_1$. Clearly $1 - f$ is a function in S_2. Then from above, we get $1 \in \overline{T(1 - f)(Y)}$ and $0 \in T(1 - f)(Y) \subseteq [0, 1]$, which show that $0 \in \overline{Tf(Y)}$ and $1 \in Tf(Y) \subseteq [0, 1]$ because $T1 = 1$ and the proof of *Claim 2* is done.

If $x \in X$ and $x \neq \min X$, then by considering f_x as in Case 1 and using a similar reasoning, we conclude that $\mathcal{I}_x \subseteq Y$. Note that, for example, if $X = [0, +\infty)$, $x = 0$, and $f_x(x) = (-x+1)\chi_{[0,1]}(x)$, we cannot find $f \in S_2$ with $coz(f_x) \cap coz(f) = \emptyset$ (compare with Case 1). Then we have to apply another method for the minimum point of X as follows:

Suppose that $x = \min X$. We first consider the case where x is a limit point of X. Define

$$f_x(z) = \frac{z - a}{x - a}\chi_{[x,a)}(z) \quad (z \in X),$$

where $a \in \mathbb{R}$ with $x < a$. Obviously, $f_x \in S_2$. Assume that there exists $y_0 \in \mathbb{R}$ such that

$$Tf_x|_{(-\infty, y_0] \cap Y} = 0 \quad and \quad Tf_x|_{(y_0, +\infty) \cap Y} \neq 0.$$

Then we can find a nonzero function $G \in AC_b(Y)$ such that $coz(G) \cap coz(T(1 - f_x)) = \emptyset$. Then we have $coz(T^{-1}G) \cap coz(1 - f_x) = \emptyset$, by Lemma 3.7. Hence $T^{-1}G(z) = 0$ for all $z \neq x$, which is impossible. This contradiction implies $Tf_x|_{[y,+\infty)\cap Y} = 0$ for some $y \in \mathbb{R}$. Especially, we get Tf_x has a compact support. Hence, as above, $\mathcal{I}_x \subseteq Y$.

Now assume that x is an isolated point of X. Let $f_x = \chi_{\{x\}}$. Then $1 - f_x = \chi_{(x,+\infty)\cap X}$. Suppose that there exists $y_0 \in \mathbb{R}$ such that $Tf_x|_{(y_0,+\infty)\cap Y} \neq 0$. Then $Tf_x = \chi_{(y_0,+\infty)\cap Y}$ and $T(1-f_x) = \chi_{(-\infty,y_0])\cap Y}$ because $Tf_x + T(1-f_x) = 1$ and $coz(Tf_x) \cap coz(T(1-f_x)) = \emptyset$. Since Y is unbounded above, we can choose $G \in AC_b(Y)$ such that $coz(G) \cap coz(T(1-f_x)) = \emptyset$ and $G \neq \alpha Tf_x$ for all $\alpha \in \mathbb{C}$. Thus $coz(T^{-1}G) \cap coz(1 - f_x) = \emptyset$, by Lemma 3.7. Since $1 - f_x = \chi_{(x,+\infty)\cap X}$, we get $T^{-1}G = \alpha_0 \chi_{\{x\}}$ for some $\alpha_0 \in \mathbb{C}$, and so $G = \alpha_0 T\chi_{\{x\}}$, which is a contradiction.

The other following cases can be obtained in a similar manner.

Case 3. X is bounded but Y is unbounded.

Case 4. X (resp. Y) is bounded below (resp. above) and unbounded above (resp. below).

Case 5. X (resp. Y) is bounded above (resp. below) and unbounded below (resp. above). □

Lemma 3.10. *Given $x \in X$, there exists a unique point $y \in Y$ such that $Tf(y) = 0$ for any $f \in AC_b(X)$ with $f(x) = 0$. Moreover, $\mathcal{I}_x = \{y\}$.*

Proof. Let $x \in X$ and $y \in \mathcal{I}_x$. Assume that $f \in AC_b(X)$ and $f(x) = 0$. We claim that $Tf(y) = 0$. Contrary to what we claim, suppose that $Tf(y) \neq 0$. Take $r > \|f\|_\infty$. Lemma 3.6 (1) allows us to choose $h \in AC_b(X)$ such that $h(x) = 1$, $0 \leq h \leq 1$ and $\||f| + rh\|_\infty = \|f \pm rh\|_\infty = r$. Notice that $|Th(y)| = 1$ because $y \in \mathcal{I}_x$. Then it follows that

$$\begin{aligned} r &= \|f \pm rh\|_\infty = \|T(f \pm rh)\|_\infty \\ &\geq |Tf(y) \pm rTh(y)| > r, \end{aligned}$$

which is a contradiction showing that $Tf(y) = 0$.

Since T^{-1} is an isometry with $T^{-1}1 = 1$, then similarly, for y, there exists $x_1 \in X$ such that $T^{-1}g(x_1) = 0$ for all $g \in AC_b(Y)$ with $g(y) = 0$. These two claims combined imply that for each $f \in AC_b(X)$ with $f(x) = 0$ we have $f(x_1) = 0$, which easily implies that $x_1 = x$ because $AC_b(X)$ separates the points of X.

Hence we have proved that y is the point in Y so that $f(x) = 0$ if and only if $Tf(y) = 0$ for any $f \in AC_b(X)$. Apparently, taking into account that $AC_b(Y)$ separates the points of Y, such y is unique. Meantime, the argument clearly yields $\mathcal{I}_x = \{y\}$. □

The above discusion allows us to define a function $\psi : X \longrightarrow Y$ such that for each $x \in X$, $\psi(x)$ is the unique point obtained in the above lemma. Indeed, $\psi(x)$ is the point with the property that

$f(x) = 0$ if and only if $Tf(\psi(x)) = 0$ for any $f \in AC_b(X)$, and we also have $\mathcal{I}_x = \{\psi(x)\}$. Meantime, it is clear that ψ is a bijective function. We set $\varphi := \psi^{-1}$.

Lemma 3.11. *For each $f \in AC_b(X)$ and $y \in Y$, $Tf(y) = f(\varphi(y))$.*

Proof. Let $f \in AC_b(X)$ and $y \in Y$. Since $(f - f(\varphi(y)))(\varphi(y)) = 0$, from Lemma 3.10, we have $T(f - f(\varphi(y)))(y) = 0$. Whence $Tf(y) = T(f(\varphi(y)))(y) = f(\varphi(y))$ since T is unital. Therefore, $Tf(y) = f(\varphi(y))$. □

Lemma 3.12. *φ is a monotonic function.*

Proof. We consider two cases based on the cardinal number of Y. If $|Y| = 2$, it is plain that φ is monotonic. Now, suppose that $|Y| > 2$. Without loss of generality, we assume that $y, y' \in Y$, $y < y'$ and $\varphi(y) < \varphi(y')$. We verify that φ is increasing (a similar argument shows that φ is decreasing if $y' < y$). Let $y_1 \in Y$. We consider the following cases:

(1) If $y < y_1 < y'$, then we claim that $\varphi(y) < \varphi(y_1) < \varphi(y')$.
(2) If $y_1 < y < y'$, then we claim that $\varphi(y_1) < \varphi(y) < \varphi(y')$.
(3) If $y < y' < y_1$, then we claim that $\varphi(y) < \varphi(y') < \varphi(y_1)$.

Contrary to what we claim in (1), let us suppose that either $\varphi(y_1) < \varphi(y) < \varphi(y')$, or $\varphi(y) < \varphi(y') < \varphi(y_1)$. Then defining

$$h(z) = \chi_{(-\infty,y]}(z) + \frac{z - y_1}{y - y_1}\chi_{(y,y_1]}(z) \quad (z \in Y),$$

or

$$h(z) = \chi_{[y',+\infty)}(z) + \frac{z - y_1}{y' - y_1}\chi_{[y_1,y')}(z) \quad (z \in Y),$$

we get $\|T^{-1}h\| \geq \mathcal{V}(T^{-1}h) > 1$ while $\|T^{-1}h\| = \|h\| = 1$, a contradiction. Thus the first claim is derived. By a similar discussion, we can deduce the other two claims. Now, it is not difficult to see that φ is increasing. Therefore, φ is a monotonic function. □

Meantime, taking into account the representation of T, it is not difficult to deduce that φ is a homeomorphism.

Now we state our main result which is obtained immediately from the previous lemmas. Let us recall here that, according to Lemma 3.1, for each $f \in AC_b(X)$, $\overline{f}$ denotes the extension of f to the closure $\bar{X}$ of X. A similar notation is used for functions in $AC_b(Y)$.

Theorem 3.13. *If $T : AC_b(X) \longrightarrow AC_b(Y)$ is a surjective linear isometry such that $T1$ is bounded away from zero, then there exist a monotonic homeomorphism $\varphi : \bar{Y} \longrightarrow \bar{X}$, and a scalar λ with $|\lambda| = 1$ such that $\overline{Tf}(y) = \lambda\overline{f}(\varphi(y))$ for all $f \in AC_b(X)$ and $y \in \bar{Y}$.*

Remark 3.14. (1) Note the surjective linear isometry T in the above result induces a homeomorphism between the closures of X and Y but not necessarily between X and Y. Indeed, since as mentioned after Lemma 3.1, the absolutely continuous functions of a set and its completion are the same, we can define a surjective linear isometry $T : AC(0,1) \longrightarrow AC[0,1]$ whereas $(0,1)$ and $[0,1]$ are not homeomorphic.

(2) It should be noted that, as the following example, borrowed from [6, Remark 4.2 (ii)], shows, that there exists a surjectve linear isometry T for which $T1$ is not bounded away from zero, and of course, T is not a weighted composition operator:

Let $X = Y = \{1, 2\}$. Define $T : AC(X) \longrightarrow AC(Y)$ by $Tf(1) = f(1)$ and $Tf(2) = f(1) - f(2)$.

However, the next result, which may be considered as a generalization of [7, Example 5] and [6, Corollary 4.4], states that if the underlying spaces are connected then $T1$ is always a unimodular function.

Corollary 3.15. *If X (or Y) is connected and $T : AC_b(X) \longrightarrow AC_b(Y)$ is a surjective isometry, then there exist a monotonic homeomorphism $\varphi : \bar{Y} \longrightarrow \bar{X}$, and a unimodular scalar λ such that $\overline{Tf}(y) = \lambda \overline{f}(\varphi(y))$ for all $f \in AC_b(X)$ and $y \in \bar{Y}$.*

Proof. We assume, without loss of generality, that Y is connected. For simplicity, set

$$\mathcal{N} = coz(T1) = \{y \in Y : T1(y) \neq 0\},$$

and $\mathcal{Z} = Y \setminus \mathcal{N}$. Clearly $\mathcal{N} \neq \emptyset$ because T is an isometry, and also $\mathcal{N}$ is an open subset of Y. Take $y_0 \in \mathcal{N}$. Choose an absolutely continuous function f on Y such that $f(y_0) = 2$, $M_f = \{y_0\}$, $\|f\|_\infty = \|f\| = 2$, $\mathcal{V}(f) \leq 1$, and $|f| \leq \frac{3}{2}$ on $Y \setminus K$ for some compact subset K of Y. An argument similar to the proof of Lemma 3.2 shows that $\mathcal{V}(T^{-1}f) \leq \|T^{-1}f\|_\infty$, which yields $\|T^{-1}f\| = \|T^{-1}f\|_\infty = 2$. Hence there is a point $x_0 \in \beta X$ such that $\widetilde{T^{-1}f}(x_0) = 2e^{i\theta}$ for some $\theta \in (-\pi, \pi]$. It is apparent that

$$3 = \|e^{i\theta}\| + \|T^{-1}f\| \geq \|e^{i\theta} + T^{-1}f\| \geq \|e^{i\theta} + T^{-1}f\|_\infty = \|e^{i\theta} + \widetilde{T^{-1}f}\|_\infty \geq |(e^{i\theta} + \widetilde{T^{-1}f})(x_0)| = 3,$$

and so $\|Te^{i\theta} + f\| = \|Te^{i\theta} + f\|_\infty = 3$. Then there exists an $y \in \beta Y$ with $|\widetilde{Te^{i\theta}}(y) + \widetilde{f}(y)| = 3$. Whence $y = y_0$ because of the equation $\|T1\| = 1$ and the properties of f. Therefore, we can deduce that $|Te^{i\theta}(y_0)| = 1$. Consequently, we can write

$$\mathcal{N} = \{y \in Y : |T1(y)| = 1\}.$$

Next, from the continuity of $T1$, it easily follows that N is a closed subset of Y. Then $\mathcal{N}$ is a non-empty clopen subset of Y. Therefore, from the connectedness of Y, we have $\mathcal{N} = Y$, which especially shows that $T1$ is a unimodular function and hence the rest of the proof follows from Theorem 3.13. □

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